BASIC PRINCIPLES OF ANALYTICAL 2D RADIAL MAGNETIC BEARING DESIGN

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Abstract

In this paper an analytical solution for the forces of a magnetic bearing is presented. The development of the equations is based on the magnetic scalar potential Φ. Since the consideration of a plain model (2D) is sufficient, we solve the Laplace equation in polar coordinates. The magnetic flux density $\mathbf{B}$ is derived from the scalar potential. The solution is a sum of a series of magnetic field waves with different orders. Only the radial and tangential components of the magnetic flux density (B_r, B_t) along a closed line surrounding the levitating object are relevant for the calculation of forces. In the next step the Maxwell stress tensor is calculated for each point on this line. This results in mechanical stress and the integration of the mechanical stress yields to the desired force. The same procedure applies for the torque in electrical machines.
First, the equation for tangential forces (which forms the torque) are derived. It was found, that only magnetic field components with the same order generate a resulting torque. More interesting for a magnetic bearing are the force generating field components. There only field waves, whose orders differ by ± 1, yield a resulting force.
Finally, the equation of forces for a magnetic bearing can be represented in an analytical way as a series of magnetic field components. These kind of representation is useful for further design and optimization considerations.

1 Introduction

Forces in magnetic bearings as well as the torque in electrical machines are determined by the magnetic field in the air gap between rotor and stator. Each point in the air gap can be assigned a mechanical stress by means of the Maxwell stress tensor. Therefore the magnetic field is evaluated along a closed line (usually a circle). The integration of the mechanical stress over this line yields a force and the torque. This is the way finite element programs (e.g. FEMM, Ansys-Maxwell) work. Since the magnetic field

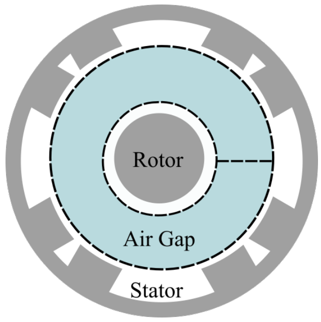

Fig. 1: Schematic view of a radial magnetic bearing with annular air gap

can have almost any shape, it makes sense to use numerical programs. On the other hand, analytic equations have the advantage that relations, between the magnetic field and the forces, become more visible. And this is the motivation of the current work.
Starting from the solution of the Laplace equation of the magnetic scalar potential Φ of an annular air gap as shown in Fig. 1, the analytic equations for the magnetic flux density are derived. In the next step the mechanical stress is formulated. Finally, an integration has to be performed to get the force and the torque.

2 Scalar Potential in Polar Coordinates

The magnetic flux density is derived from a scalar potential Φ:

$$\mathbf{B}(r,\varphi) = \mu_0 \cdot \text{grad}\,\Phi(r,\varphi) \tag{1}$$

For the annular airgap polar coordinates are used, hence there is a radial component B_r and a tangential component B_t of the magnetic flux density:

$$B_r = \mu_0 \cdot \frac{\partial \Phi}{\partial r} \tag{2}$$

$$B_t = \mu_0 \cdot \frac{1}{r} \cdot \frac{\partial \Phi}{\partial \varphi} \tag{3}$$

Since scalar potential is only unique defined in simply connnected areas an artifical boundary in the airgap with certain boundary conditions is introduced (see the horizontal dashed line in Fig. 1). At this boundary the following boundary conditions are defined to guarantee periodicity in tangential direction:

$$B_r(r,0) = B_r(r,2\pi) \tag{4}$$

$$B_t(r,0) = B_t(r,2\pi) \tag{5}$$

Since the magnetic flux density needs to fulfil second Maxwell Law div **B**= 0, we get straightforward the following Laplace equation in polar coordinates:

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 \Phi}{\partial \varphi^2} = 0 \tag{6}$$

For the solution of the Laplace equation (6) the method of separation of variables is used. This leads to two independent differential equations:

$$\frac{\partial^2 F}{\partial \varphi^2} + m^2 \cdot F = 0 \tag{7}$$

$$r^2 \cdot \frac{\partial^2 R}{\partial r^2} + r \cdot \frac{\partial R}{\partial r} - m^2 \cdot R = 0 \tag{8}$$

The first one is the well kown equation for undamped vibrations, while the second one is Cauchy-Euler equation. The detailed solution is shown in [1] and [2]. For $m > 0$ we get:

$$F_m(\varphi) = a_m \cdot \cos(m\varphi) + b_m \cdot \sin(m\varphi) \tag{9}$$

$$R_m(r) = c_m \cdot r^m + d_m \cdot r^{-m} \tag{10}$$

And for $m = 0$:

$$F_0(\varphi) = a_0 + b_0 \cdot \varphi \tag{11}$$

$$R_0(r) = c_0 + d_0 \cdot \ln(r) \tag{12}$$

The complete solution for the magnetic scalar potential is:

$$\Phi(r,\varphi) = (c_0 + d_0 \cdot \ln(r)) \cdot (a_0 + b_0 \cdot \varphi) + \sum_{m=1}^{\infty} c_m r^m \left[a_m \cos(m\varphi) + b_m \sin(m\varphi)\right] + \sum_{m=1}^{\infty} \frac{d_m}{r^m} \left[a_m \cos(m\varphi) + b_m \sin(m\varphi)\right] \tag{13}$$

In Eq.13 are two sums, which look very similar to the result of a fourier transformation. Additionaly, there is a simple radial dependence of the potential. The terms in the first sum of the potential increase with $\sim r^m$, in the second sum the terms decrease proportionally with $\sim r^{-m}$. This second relation is nothing else than the equation for a 2m-pole in the plain model. For example, the equation for an dipole (m=1) is according to [3]:

$$\Phi = \frac{d}{r} \left[a \cos(\varphi) + b \sin(\varphi)\right] \tag{14}$$

The corresponding potential is shown in Fig. 2 For the next order $m = 2$ we get the potential for

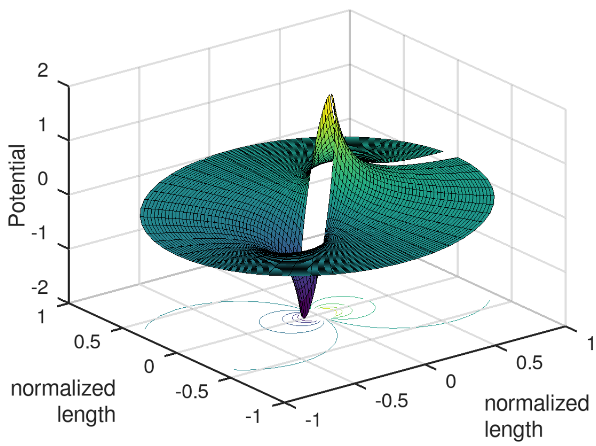

Fig. 2: Potential of a dipole ($m = 1$)

a quadrupole, shown in Fig. 3. The first sum in Eq.13 gives a increasing potential with increasing radius. In case of $m = 2$ (quadrupole) we

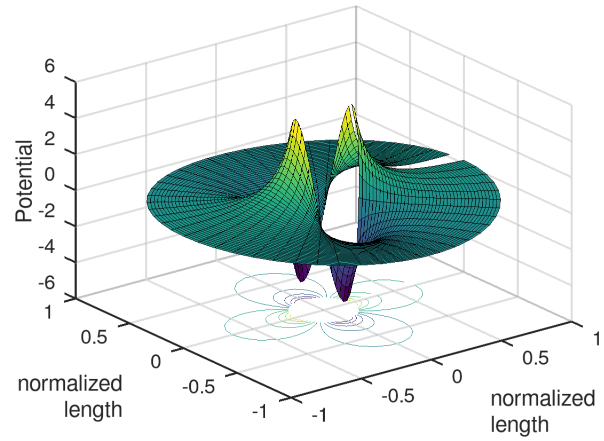

Fig. 3: Potential of a quadrupole ($m = 2$)

have a proportionalitiy of $\Phi \sim r^2$ (see Fig. 4), which is the inverse of the first term. So we will call it the inverse quadrupole or generally speaking: the inverse m-pole.

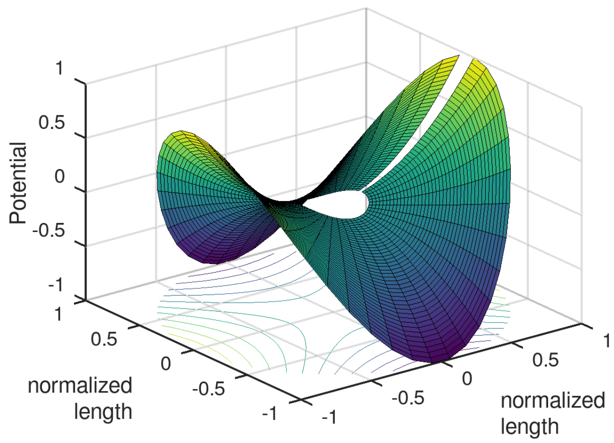

Fig. 4: Potential of an inversed quadrupole ($m = 2$)

Both, a particular m-pole and the inverse m-pole have the same factors a_m, b_m in Eq.13. These factors correspond to a phase, that implies they have the same direction in space. But depending on the actual boundary conditions, a potential of m-th order can be a superposition of m-poles and inverse m-poles of any phase. It can be shown, that any superposition of potentials of m-th order can be again separated in two sums, one sum containing the m-pole, the other on containing all inverse m-poles. For the sake of limited space, we refrain from showing the proof, but we introduce a different notation for the potential, which is shown in Eq. 15.
In the follwing also the zero terms F_0 and R_0 from Eq.13 are neglected. Even though the case $b_0 \neq 0; d_0 = 0$ describes the well known case of the magnetic field around a single current carrying wire, we are convinced that these cases are minor important.

3 Maxwell stress tensor in 2D

The gradient of the potential yields the flux density (Eq. 21 and 22). With the known field components, each point in space can be assigned a mechanical stress. The equations for the radial stress σ_r and the tangential stress σ_r for the plain model are according to [4]:

$$\begin{pmatrix} \sigma_r \\ \sigma_t \end{pmatrix} = \frac{1}{2\mu_0} \begin{pmatrix} B_r^2 - B_t^2 \\ 2 \cdot B_r B_t \end{pmatrix} \tag{20}$$

The integration over the tangential stress σ_t results in the torque.

$$\Phi(r, \varphi) = \sum_{m=1}^{\infty} r^m \cdot C_m \cdot \sin(m\varphi + \alpha_{mc}) + \sum_{m=1}^{\infty} r^{-m} \cdot D_m \cdot \sin(m\varphi + \alpha_{md}) \tag{15}$$

$$C_m = \sqrt{(c_{m1}a_{m1} + c_{m2}a_{m2})^2 + (c_{m1}b_{m1} + c_{m2}b_{m2})^2} \tag{16}$$

$$D_m = \sqrt{(d_{m1}a_{m1} + d_{m2}a_{m2})^2 + (d_{m1}b_{m1} + d_{m2}b_{m2})^2} \tag{17}$$

$$\alpha_{mc} = \arctan \frac{c_{m1}a_{m1} + c_{m2}a_{m2}}{c_{m1}b_{m1} + c_{m2}b_{m2}} \tag{18}$$

$$\alpha_{md} = \arctan \frac{d_{m1}a_{m1} + d_{m2}a_{m2}}{d_{m1}b_{m1} + d_{m2}b_{m2}} \tag{19}$$

$$B_r(r,\varphi) = \mu_0 \sum_{m=1}^{\infty} m \cdot C_m \cdot r^{m-1} \cdot \sin(m\varphi + \alpha_{mc}) - \mu_0 \sum_{m=1}^{\infty} m \cdot \frac{1}{r^{m+1}} \cdot D_m \cdot \sin(m\varphi + \alpha_{md}) \tag{21}$$

$$B_t(r,\varphi) = \mu_0 \sum_{m=1}^{\infty} m \cdot C_m \cdot r^{m-1} \cdot \cos(m\varphi + \alpha_{mc}) + \mu_0 \sum_{m=1}^{\infty} m \cdot D_m \cdot \frac{1}{r^{m+1}} \cdot \cos(m\varphi + \alpha_{md}) \tag{22}$$

$$\begin{aligned} B_rB_t(r,\varphi) = \mu_0^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m \cdot n \cdot [& C_mC_n \cdot r^{+m+n-2} \cdot \sin(m\varphi + \alpha_{mc}) \cdot \cos(n\varphi + \alpha_{nc}) \ldots \\ & + C_mD_n \cdot r^{+m-n-2} \cdot \sin(m\varphi + \alpha_{mc}) \cdot \cos(n\varphi + \alpha_{nd}) \ldots \\ & - D_mC_n \cdot r^{-m+n-2} \cdot \sin(m\varphi + \alpha_{md}) \cdot \cos(n\varphi + \alpha_{nc}) \ldots \\ & - D_mD_n \cdot r^{-m-n-2} \cdot \sin(m\varphi + \alpha_{md}) \cdot \cos(n\varphi + \alpha_{nd}) \,] \end{aligned} \tag{23}$$

For the force a further transformation into cartesian system needs to be done:

$$\sigma_x = \sigma_r \cos\varphi - \sigma_t \sin\varphi \tag{24}$$

$$\sigma_y = \sigma_r \sin\varphi + \sigma_t \cos\varphi \tag{25}$$

Solving this equations (20 - 25) requires the calculation of the products of radial and tangential components of the magnetic flux density. Here as an example only the product B_rB_t is shown in eq. 23.

4 Torque

Integration along a closed line (2D) yields a force, here an radial and a tangential force. The latter yields the torque in an electrical machine:

$$T = \frac{1}{2\mu_0} r l_a \int_0^{2\pi} 2 \cdot B_r \cdot B_t r d\varphi. \tag{26}$$

l_a stands for the axial direction. This integral contains the product of both field components (B_r, B_t), resulting in a double sum (see Eq.23). We don't want to show the details, rather explain the principles. In the expression are products of type:

$$\sin(m\varphi + \alpha_{mc}) \cdot \sin(n\varphi + \alpha_{nd}) \tag{27}$$

These kind of products can be splitted into two sums using an addition theorem [5]:

$$\begin{aligned} & 0.5 \cdot \cos([m-n]\,\varphi + \alpha_{mc} - \alpha_{nd}) - \\ & 0.5 \cdot \cos([m+n]\,\varphi + \alpha_{mc} + \alpha_{nd}) \end{aligned} \tag{28}$$

These sums can be easy integrated and yield due to the integration limits (0,2π) always zero, except in the case $m = n$. After performing all the integration steps we get for the torque:

$$T = \frac{2\pi r l_a}{\mu_0} \sum_{m=0}^{\infty} \frac{m^2}{r^2} C_m D_m \sin(\alpha_{mc} - \alpha_{md}) \tag{29}$$

This equation (29) shows, that only field waves of the same order yields a contribution to the torque. Further the equation contains no products of type C_m^2 and D_m^2 any more. That means only m-poles and the according inverse m-poles contributes to the torque. Simply spoken: One dipol itself creates no torque - second inverse dipole is necessary.

5 Forces

For the calculation of forces the following integrals needs to be solved:

$$F_x = \frac{1}{2\mu_0} \int_0^{2\pi} [\sigma_r \cos(\varphi) - \sigma_t \sin(\varphi)]\, r d\varphi \tag{30}$$

$$F_y = \frac{1}{2\mu_0} \int_0^{2\pi} [\sigma_r \sin(\varphi) + \sigma_t \cos(\varphi)]\, r d\varphi \tag{31}$$

We show the solutions steps using the force F_y, the steps for F_X are analogous. In terms of field components, the equation is as follows:

$$F_y = \frac{r}{2\mu_0}\int_0^{2\pi}\left[\left(B_r^2 - B_t^2\right)\sin(\varphi)\right]d\varphi \quad (32)$$
$$-\frac{r}{2\mu_0}\int_0^{2\pi}\left[2\cdot B_r\cdot B_t\cos(\varphi)\right]d\varphi \quad (33)$$

Here, contrary to the torque, 3 double sums have to be solved: B_r^2, B_t^2 and B_rB_t. These double sums contain products of following type:

$$\cos\left(m\varphi+\alpha_{mc}\right)\cos\left(n\varphi+\alpha_{nd}\right)\sin(\varphi) \quad (34)$$

These kind of products can be again splitted into sums using an addition theorems [5], the result is shown in eq 35:

$$\begin{aligned}&+\frac{1}{4}\sin\left(\left[-m+n+1\right]\varphi-\alpha_{mc}+\alpha_{nc}\right)\\&+\frac{1}{4}\sin\left(\left[+m-n+1\right]\varphi+\alpha_{mc}-\alpha_{nc}\right)\\&+\frac{1}{4}\sin\left(\left[-m-n+1\right]\varphi-\alpha_{mc}-\alpha_{nc}\right)\\&+\frac{1}{4}\sin\left(\left[+m+n+1\right]\varphi+\alpha_{mc}+\alpha_{nc}\right)\end{aligned} \quad (35)$$

These sums again can be easy integrated and yield due to the integration limits (0,2π) always zero, except in the case $n = m \pm 1$. Because of these special cases we split the mentioned double sum as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} = \sum_{m=1}^{\infty}\sum_{\substack{n=1\\n\neq m\pm1}}^{\infty} + \sum_{\substack{m=1\\n=m+1}}^{\infty} + \sum_{\substack{m=2\\n=m-1}}^{\infty} \quad (36)$$

The first term will be equal to zero. The following expressions only have to be summed over m. The last sum starts with $m = 2$, since the sum over n starts at 1 and $n = 0$ not exist.

$$\begin{aligned}F_y = \mu_0 &\sum_{m=1}^{\infty} m(m+1)\frac{\pi}{2}\dots\\&\left(+r^{2m}C_mC_{m+1}\sin(\alpha_{mc}-\alpha_{m+1c})\right.\\&-r^{-2}C_mD_{m+1}\sin(\alpha_{mc}-\alpha_{m+1d})\\&-r^{0}C_{m+1}D_m\sin(\alpha_{md}-\alpha_{m+1c})\\&\left.-r^{-2m-2}D_mD_{m+1}\sin(\alpha_{md}-\alpha_{m+1d})\right)\\&+\mu_0\sum_{m=2}^{\infty} m(m-1)\frac{\pi}{2}\dots\\&\left(+r^{2m-2}C_mC_{m-1}\sin(\alpha_{mc}-\alpha_{m-1c})\right.\\&-r^{0}C_mD_{m-1}\sin(\alpha_{mc}-\alpha_{m-1d})\\&-r^{-2}C_{m-1}D_m\sin(\alpha_{md}-\alpha_{m-1c})\\&\left.-r^{-2m}D_mD_{m-1}\sin(\alpha_{md}-\alpha_{m-1d})\right)\end{aligned} \quad (37)$$

This expression can further reduced, by starting the second sum at $m = 1$ again. Therefore in the second sum all m need to be replaced by $m+1$.

$$\begin{aligned}F_y = \mu_0 &\sum_{m=1}^{\infty} m(m+1)\frac{\pi}{2}\dots\\&\left(r^{2m}C_mC_{m+1}\sin(\alpha_{mc}-\alpha_{m+1c})\right.\\&+r^{-2}C_mD_{m+1}\sin(\alpha_{mc}-\alpha_{m+1d})\\&-r^{0}C_{m+1}D_m\sin(\alpha_{md}-\alpha_{m+1c})\\&-r^{-2m-2}D_mD_{m+1}\sin(\alpha_{md}-\alpha_{m+1d})\\&-r^{2m}C_{m+1}C_m\sin(-\alpha_{m+1c}+\alpha_{mc})\\&-r^{0}C_{m+1}D_m\sin(-\alpha_{m+1c}+\alpha_{md})\\&+r^{-2}C_mD_{m+1}\sin(-\alpha_{m+1d}-\alpha_{mc})\\&\left.+r^{-2m-2}D_{m+1}D_m+\sin(-\alpha_{m+1d}+\alpha_{md})\right)\end{aligned} \quad (38)$$

This can be further reduced and finally, we get for the force:

$$\begin{aligned}F_y = \mu_0 &\sum_{m=1}^{\infty} m(m+1)\frac{\pi}{2}\dots\\&\left(-C_{m+1}D_m\sin\left[\alpha_{md}-\alpha_{m+1c}\right]\right.\\&\left.+r^{-2}C_mD_{m+1}\sin\left[\alpha_{mc}-\alpha_{m+1d}\right]\right)\end{aligned} \quad (39)$$

This equation (39) shows, that only the combination of a m-pole (C_m) and an inverse m-pole ($D_{m\pm1}^2$) results in a force contribution. The m-poles itself deliver no contribution, as well as the inverse m-poles. Despite to the torque, the field waves need to be of ±1 order.

6 Conclusions

The analytical solution for the forces of a magnetic bearing and the torque of an electrical machine is presented. It was shown, that the magnetic field can be represented as a superpositon of m-poles and inverse m-poles. This means that any magnetic field in the annular air gap can be decomposed into these components. While for the torque only field waves of the same order ($n = m$) yield a contribution, for the forces field waves, whose orders differ by $n = m \pm 1$, yield a resulting force.

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