Feedback Linearization Applied to a MIMO Active Magnetic Bearing: A Systematic Approach and Application to a Balance Beam

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Abstract

This paper illustrates a systematic approach to feedback linearization for a magnetic bearing system. This method relies upon the construction of a nonlinear coordinate transformation that insures a unique feedback linearization law exists which has several practical properties: the linearizing feedback is guaranteed regular; the resulting linearized system is guaranteed controllable; and beam angular position regulation is achievable. Moreover, the construction of a diffeomorphism allows the control engineer to assert operating constraints on the actuator fluxes. In particular, this paper investigates constant flux sum and constant product constraints and compares their relative merits.

1 Introduction

Magnetic bearings are nonlinear devices, but normally they are modeled using Jacobian linearization about a fixed operating point (of both position and magnetization) [6]. However, there are many instances when the bearing system must operate away from the linearization neighborhood. For example, magnetic bearings typically have large clearances and in some cases the rotor must operate reliably over the entire clearance space. A particular application where this is true is the magnetic bearing suspended impeller for an artifical heart pump, which does not have back-up bearings and must sometimes operate near the bearing clearance limit [9]. Similarly, actuator fluxes might also deviate far from nominal conditions, compromising both performance and stability. Feedback linearization avoids these problems by providing an exact linearization over the entire operating clearance of the actuator.

On the other hand, many applications require a minimum control effort, while still achieving a specified performance [3]. One objective is to keep the bias flux as low as possible to

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limit coil current induced ohmic losses. However, the flux ϕ governs the force slew rate $\dot{F}$ of the actuator, placing restrictions on bearing performance. Mathematically,

$$\dot{F} \propto \frac{d}{dt}\phi^2 \propto \phi\frac{v}{N} \tag{1}$$

where N and v are the coil turns and voltage respectively. Therefore, with decreased bias fluxes, the actuator's dynamic capacity gradually diminishes until it is zero at the origin. Feedback linearization provides a method of constraining actuator fluxes independently of rotor position, which allows the control designer to implement low power biasing strategies.

Several works have considered solutions to the above mentioned difficulties. Levine *et. al.* [10] employed a polynomial to define the flux constraints on a magnetic bearing in order to keep the bias flux low but still avoid zero bias flux and the resulting loss of stability. Kim and Kim [14] explored the use of gain scheduled controllers over a range of actuator motion. Trumper *et. al.* [5] applied feedback linearization to the magnetic suspension of a ball with resulting improvement in tracking and stability away from the nominal operating point. A globally linearized current control law in controllability canonical form was developed which was linear with ball position and proportional to the square root of the acceleration to be applied to the mass. Mittal and Menq [15] investigated the use of a geometric feedback linearization technique about the origin for a suspended ball similar to that of [5]. They noted that their method is subject to uncertainty errors due to parameter variations and external disturbances but that these effects can be overcome with good nonlinear control design methods with resulting robust control. The conventional linearized model destabilizes under certain conditions that the feedback linearized controller can still control, such as large ball motions. Lindlau [4] investigated a feedback linearization approach to dynamic biasing. Recently, Li [12] as well as Li and Mao [13] discussed exact linearization using several different control configurations: constant voltage sum (CVS), constant current sum (CCS) and constant flux sum (CFS). They showed better tracking properties of the magnetic bearing with these control algorithms.

While several authors have discussed exact linearization, no systematic method of developing the approach has been presented in the literature for MIMO magnetic bearings to date. This paper discusses a very general special coordinate transformation developed by Nijmeijer [8] and Isidori [1] that can be used to evaluate various control strategies for magnetic bearings and applies this method to some particular magnetic bearing systems. Slotine and Li [11] discuss a similar approach. It is well known that an infinite number of transformations can be found to linearize a nonlinear system. However, a very specific one must be constructed, called a diffeomorphism, that twists the original system into a new one with specific properties: unique feedback linearization that is regular and guarantees controllability. This mathematical technique provides an unambiguous way of obtaining the coordinate transformation and a natural way of placing constraints on the actuator fluxes.

2 System Description

Figure 1 illustrates the system to be considered in this paper. Essentially, a rigid beam of moment J is simply supported at its center of mass by a pivot designated by O. At

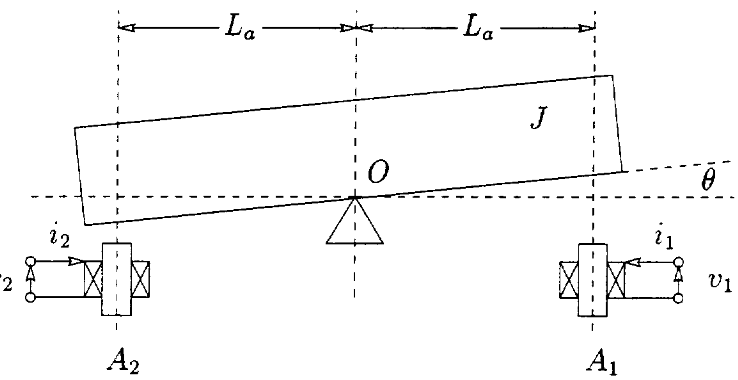

Figure 1: Symmetric balance beam with magnetic bearings.

a length L_a to either side of the pivot are horse-shoe electromagnets labeled $A1$ and $A2$, which produce forces F_1 and F_2. Each actuator has a voltage input v_1 and v_2 and flux state variables ϕ_1 and ϕ_2. The remaining two states of the system are the beam angle, θ, and angular velocity, $\dot{\theta}$. The system can be adequately described by the following differential equation

$$J\ddot{\theta} = \tau_m(\phi, \theta) + d(t) \tag{2}$$

where $\tau_m(\phi, \theta)$ is the magnetic actuator torque and $d(t)$ is an unknown external moment.

We now calculate the function $\tau_m(\phi, \theta)$ assuming the coil voltage as input. Begin by noting the that the flux linkage, λ, and flux , ϕ, are related by the number of wire turns, N, in the actuator coil

$$\lambda = N\phi$$

Then from Faraday's Law we have that

$$v = \frac{d\lambda}{dt} + Ri = N\frac{d\phi}{dt} + Ri \tag{3}$$

where R and i are the coil resistance and current respectively. However, Ampere's Law gives the current, i, in terms of flux ϕ, as

$$i = \frac{1}{N}\mathcal{R}(\theta)\phi$$

where the actuator reluctance, $\mathcal{R}(\theta)$, for actuators $A1$ and $A2$ is

$$\mathcal{R}_{\substack{1\\2}}(\theta) = \frac{2(g_0 \pm L_a\theta)}{\mu_0 A}$$

Here g_0 and A are the nominal air gap length and air gap area, respectively. When combined with Eq. (3) this gives

$$v_1 = N\frac{d\phi_1}{dt} + \frac{2R}{\mu_0 NA}(g_0 + L_a\theta)\phi_1 \quad \text{and} \quad v_2 = N\frac{d\phi_2}{dt} + \frac{2R}{\mu_0 NA}(g_0 - L_a\theta)\phi_2 \tag{4}$$

which expresses the relationship from voltage to actuator flux. A corresponding force between the stator and rotor components develops in response to the flux between them. This relationship between force and flux can be approximated by evaluating the magnetic energy present within the actuator given variations in beam displacement. Neglecting magnetic fringing and leakage the magnetic energy stored in the actuator is [2]

$$E(\theta) = \frac{1}{2}\mathcal{R}(\theta)\phi^2$$

Differentiation of the reluctance with respect to beam angle provides the force expression

$$F = -\frac{\partial E}{\partial \theta} = -\frac{1}{2}\frac{\partial \mathcal{R}}{\partial \theta}\phi^2$$

Then, given a torque arm of length L_a, the actuator torque for $A1$ and $A2$ becomes

$$\tau_{\frac{1}{2}} = F_{\frac{1}{2}} \cdot L_A = \mp \frac{L_a}{\mu_0 A}\phi_{\frac{1}{2}}^2 \cdot L_a \tag{5}$$

Finally, in state space form the nonlinear dynamic equations describing the magnetic bearing system become

$$\frac{d}{dt}\left\{\begin{array}{c}\phi_1\\ \phi_2\\ \theta\\ \dot{\theta}\end{array}\right\} = \left\{\begin{array}{c}-c_2(g_0+L_a\theta)\phi_1\\ -c_2(g_0-L_a\theta)\phi_2\\ \dot{\theta}\\ c_1(\phi_2^2-\phi_1^2)\end{array}\right\} + \left\{\begin{array}{c}\frac{1}{N}\\ 0\\ 0\\ 0\end{array}\right\}v_1 + \left\{\begin{array}{c}0\\ \frac{1}{N}\\ 0\\ 0\end{array}\right\}v_2 \tag{6}$$

$$\frac{dx}{dt} = f(x) + g_1 v_1 + g_2 v_2 \tag{7}$$

For convenience we have defined $c_1 = \frac{L_a^2}{J\mu_0 A}$ and $c_2 = \frac{2R}{\mu_0 N^2 A}$. This system has two inputs and at most two outputs. One output will be the beam angle $h_1 = \theta$. The other, h_2, will not be chosen until later because its selection will profoundly affect the construction of a diffeomorphism that makes feedback linearization possible . As stated earlier the diffeomorphism twists the nonlinear system described by (7) so that its nonlinear components can be algebraically canceled. Although one might construct a feedback linearization in an "ad hoc" approach there is no guarantee that the resulting linear system would be controllable and implementable. In the next section the controllability of system (7) will be investigated to prove that the system is feedback linearizable.

3 Nonlinear Controllability

This section will be to show that the system exhibits the properties of strong accessibility and feedback invariance. Conceptually, we are interested in the overlap of the input space and the state space defined by $f(x)$. If the overlap is the same dimension as the state space then some type of controllability exists for the system (7). To explore this concept further, define $\mathcal{C}_0$ as the smallest Lie algebra which contains g_1 and g_2 and also satisfies $[f, X] \in \mathcal{C}_0$ for all $X \in \mathcal{C}_0$. Using the above definition we generate a corresponding distribution

$$C_0(x) = \text{span}\{X(x)|X\text{vector field in } \mathcal{C}_0\}$$

In the literature, $\mathcal{C}_0$ is the *accessibility algebra* and C_0 the *accessibility distribution.* With these definitions we posit the following

Theorem 1 *The system (7) is said to have the property of strong accessibility from the point x_e if*

$$dim\, C_0(x_e) = n$$

where n is the system order.

Readers interested in the proof of the above theorem are referred to [8]. Note that for linear systems this reduces to the Kalman Rank Condition. For the system (7), g_1 and g_2 are constant, therefore the distribution spanned by the vector fields g_1 and g_2 is trivially involutive [1]. so that the accessibility distribution for the system does not contain brackets of g_1 and g_2. Hence, the accessibility distribution for system (7) becomes

$$C_0(x_e) = \text{span}\,\{g_1, g_2, [f, g_1], [f, g_2], [f, [f, g_1]], [f, [f, g_2]], [f, [f, [f, g_1]]], [f, [f, [f, g_2]]], ...\} \quad (8)$$

where the maximum *possible* dimension of C_0 is the dimension of the state space described by (7), in this case four. Examination of C_0 shows that we need only take

$$C_0(x_e) = \text{span}\,\{g_1, g_2, [f, g_1], [f, g_2], [f, [f, g_1]], [f, [f, g_2]]\} \quad (9)$$

to achieve a dimension of four because additional vector combinations are redundant. Given this information, we select Eq. (9) as the definition of the accessibility distribution for this particular system. Interestingly, strong accessibility vanishes when the equilibrium point of the system is chosen to be $x_e = (0,0,0,0)$, which agrees with intuition based on Eq. (1). However, for an equilibrium point in the open set $\mathbb{R}^4 - \mathbf{0}$, say $x_e = (\epsilon, \epsilon, 0, 0)$, $\dim C_0(x_e) = 4$ so that the system exhibits strong accessibility by Theorem 1. This suggests that the system described by (7) must have at least a very small bias flux to be controllable.

Now we may use the following theorem of Nijmeijer [8] to determine feedback linearizability.

Theorem 2 *Given the system $\dot{x} = f(x) + g_1 u_1 + g_2 u_2$ is strongly accessible in x_e and $f(x_e) = 0$ the system is feedback linearizable [2] if and only if the distributions*

$$D_k(x) = span\{ad_f^r g_1, ..., ad_f^r g_2 | r = 0, \ldots, k-1\}, k = 1, 2, 3, 4$$

[1]A distribution G is involutive if the lie bracket $[\beta_1, \beta_2]$ of any pair of vectors fields β_1 and β_2 belonging to G is a vector field belonging to G. Mathematically,

$$\beta_1 \in G,\, \beta_2 \in G \Rightarrow [\beta_1, \beta_2] \in G$$

[2]Briefly, $\dot{x} = f(x) + g(x)v$ is feedback linearizable if the following statements are true:

(i) a diffeomorphism, $z = \Phi(x)$ is defined around x_e

(ii) regular feedback is defined

such that the nonlinear system $\dot{x} = f(x) + g(x)v$ can be written in the form $\dot{z} = Az + Bu$

are involutive and of constant dimension in the neighborhood x_e [3]. *Furthermore, the resulting linear system is controllable.*

In the case of system (7), the distributions are in fact all involutive, thus satisfying the first condition of Theorem 2. The next condition is that the dimensions of these distributions are constant (i.e. feedback invariant). This test is performed by evaluating the dimension of the distributions at x and at $x + \delta$, where δ is a state perturbation. Performing this test shows the distributions D_1, D_2, D_3, D_4 have invariant dimensions $2, 3, 4, 4$ respectively, and so satisfy the second condition of Theorem 2. As with controllability, the system fails the feedback invariance criterion at an equilibrium point $x_e = (0, 0, 0, 0)$ such that the distributions D_1, D_2, D_3, D_4 have invariant dimensions $2, 3, 3, 3$. This further emphasizes the ill-defined nature of the system at the origin (without bias).

4 Construction of Diffeomorphism

In the previous section system (7) was shown to be feedback linearizable. Consequently, a coordinate transformation $z = \Phi(z)$ exists such that its application will twist the original system into Byrnes-Isidori normal form. Here $\Phi(x)$ represents a function in $\mathbb{R}^n$ and is called a *global diffeomorphism* when the following conditions are satisfied:

(i) $\Phi^{-1}(z)$ exists such that $\Phi^{-1}(\Phi(x)) = x$ (invertible)

(ii) $\Phi(x)$ and $\Phi^{-1}(z)$ both have continuous partial derivatives of any order (smooth mappings).

Application of the diffeomorphism to the description of the nonlinear system can be calculated as

$$\begin{aligned} \dot{z} &= \hat{f}(z) + \hat{g}(z)v \\ y &= \hat{h}(z) \end{aligned}$$

where

$$\hat{f}(z) = \frac{d\Phi}{dx} f(x)\bigg|_{x=\Phi^{-1}(z)} \qquad \hat{g}(z) = \frac{d\Phi}{dx} g(x)\bigg|_{x=\Phi^{-1}(z)} \qquad \hat{h}(z) = h(x)|_{x=\Phi^{-1}(z)} \tag{10}$$

We wish to construct a diffeomorphism that twists the nonlinear system into a specific form (i.e. normal form). Let us begin by considering the normal form of the system described by Eq. (7) given relative degree $r = 3, 1$

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= \beta_{11}(z)v_1 + \beta_{12}(z)v_2 + \alpha_1(z) \\ \dot{z}_4 &= \beta_{21}(z)v_1 + \beta_{22}(z)v_2 + \alpha_2(z) \end{aligned} \tag{11}$$

[3]A convenient shorthand notation for the k-th recursive bracketing of f with g is $ad_f^k g(x) = [f, [f, \ldots [f, g] \ldots]]$

Let us elaborate on each equation of system (11) beginning with the uppermost

$$\dot{z}_1 = \frac{d\Phi_1}{dt} = \frac{\partial \Phi_1}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial \Phi_1}{\partial x}(f(x) + g_1(x) + g_2(x)) = z_2$$

and then set the first channel of the coordinate transformation to be the first output, mathematically

$$\Phi_1 = z_1 = h_1 \qquad (12)$$

where h_1 is the first output. Then the previous equation for $\dot{z}_1$ becomes

$$\dot{z}_1 = L_f h_1 + L_{g_1} h_1 + L_{g_2} h_1$$

Here $L_f(\cdot)$ is the Lie derivative [4]. Now, inspection of the system (11) shows that the input should not appear in the first equation. Therefore, the following must be true

$$L_{g_1} h_1 = 0 \quad \text{and} \quad L_{g_2} h_1 = 0$$

so that

$$\dot{z}_1 = L_f h_1 = z_2 \qquad (13)$$

Continuing onto the second equation of system (11) and using the fact that $z_2 = \Phi_2 = L_f h_1$ we have

$$\begin{aligned} \dot{z}_2 = \frac{d\Phi_2}{dt} = \frac{\partial \Phi_2}{\partial x}\frac{\partial x}{\partial t} &= L_f h_1(f(x) + g_1(x) + g_2(x)) \\ &= L_f^2 h_1 + L_f L_{g1} h_1 + L_f L_{g2} h_1 = z_3 \end{aligned}$$

Once again, inspection of the system (11) shows that the input should not appear yet. As such

$$L_f L_{g_1} h_1 = 0 \quad \text{and} \quad L_f L_{g_2} h_1 = 0$$

then

$$\dot{z}_2 = L_f^2 h_1 = z_3 \qquad (14)$$

in similar fashion $\dot{z}_3$ is calculated

$$\begin{aligned} \dot{z}_3 = \frac{d\Phi_3}{dt} = \frac{\partial \Phi_3}{\partial x}\frac{\partial x}{\partial t} &= L_f^2 h_1(f(x) + g_1(x) + g_2(x)) \\ &= L_f^3 h_1 + L_f^2 L_{g_1} h_1 + L_f^2 L_{g_2} h_1 \end{aligned}$$

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[4]This differential operation $L_f\lambda(x)$ is called the Lie derivative of λ along f where λ is a real-valued function and f a vector field and is calculated as

$$\langle d\lambda(x), f(x)\rangle = \frac{\partial \lambda}{\partial x} f(x) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(x)$$

where product of the Lie derivative is a new real-valued function.

Since the output must appear in this equation to satisfy the normal form we have that

$$L_f^2 L_{g_1} h_1 \neq 0 \quad \text{and} \quad L_f^2 L_{g_2} h_1 \neq 0$$

Having exhausted the output h_1 by this process, we proceed to calculate z_4 using the output h_2. The results of Isidori [1] explain that if the number of differentiations before an input appears is r for each output h such that $r_1 + \ldots + r_m = n$ (where there are m inputs and the system is n^{th} order), then a coordinate transformation exists that is both nonsingular and twists the nonlinear system the into normal form. Thus, since the first output was differentiated three times before the input was allowed to appear, $r_1 = 3$, we desire to differentiate the second output only once before the input may appear, $r_2 = 1$. From a geometric perspective, we desire to maximize the dimension of the system manifold such that the zero dynamics sub-manifold must be 0-dimension. The zero dynamics manifold represents the "internal" dynamics of the system which for specific input and initial conditions constrains the output to be identically zero. For linear systems, the zero dynamics are those eigenvalues of the system which coincide with the zeros of the transfer function. If possible, we avoid zero dynamics by proper selection of the output. Hence we select the fourth channel of the coordinate transformation to be

$$\Phi_4 = z_4 = h_2 \tag{15}$$

and since this is the last equation of the system, the input must appear so that $r_1 + r_2 = 4 = n$

$$\dot{z}_4 = \frac{d\Phi_4}{dt} = \frac{\partial \Phi_4}{\partial x}\frac{\partial x}{\partial t} = L_f h_2 + L_{g_1} h_2 + L_{g_2} h_2$$

and thus

$$L_{g_1} h_2 \neq 0 \quad \text{and} \quad L_{g_2} h_2 \neq 0$$

Several conditions were placed on equations for $\dot{z}_1$ through $\dot{z}_4$ to ensure that the coordinate transformation results in normal form with no zero-dynamics. Summarizing

$$L_{g_j} L_f^k h_i(x) = 0 \tag{16}$$

for all $1 \leq j \leq m$, for all $1 \leq i \leq m$, and for all $k \leq r_i - 1$. In addition, the following matrix must be nonsingular:

$$\begin{bmatrix} L_{g_1} L_f^2 h_1 & L_{g_2} L_f^2 h_1 \\ L_{g_1} h_2 & L_{g_2} h_2 \end{bmatrix} \tag{17}$$

The above conditions are in fact the definition of the *relative degree* $\{r_1, r_2\}$ of this system[1].

Although the framework for building a coordinate transformation that twists the nonlinear system described by (7) has been provided by the previous analysis, i.e.

$$\begin{aligned} \Phi_1 &= h_1 \\ \Phi_2 &= L_f h_1 \\ \Phi_3 &= L_f^2 h_1 \\ \Phi_4 &= h_2 \end{aligned}$$

the outputs h_1 and h_2 have not been chosen. It is clear that the conditions (16)-(17) place restrictions on the choice of output such that normal form can be achieved. Consider the primary control objective for the system, which is that $\theta(t) \to 0$ as $t \to \infty$. As such, one output must be θ. Recall that the first output, θ, must be differentiated three times before an input appears. To satisfy the condition $r_1 + r_2 = n = 4$, we must only differentiate the second output once before the input appears. Inspection of system (7) shows that *any* smooth function $F(\phi_1, \phi_2)$ satisfies this condition. In this paper two functions for h_2 will be considered: $h_2 = (\phi_1 + \phi_2)/2$ and $h_2 = \phi_1\phi_2$. The first choice offers the possibility of controlling the nominal flux dynamically, while the second dismisses the concept of biasing altogether in favor of complimentary flux control. Either choice satisfies the conditions of relative degree imposed by conditions (16)-(17) at the equilibrium of the system, $x_e = (\epsilon, \epsilon, 0, 0)$ where $\epsilon > 0$.

5 Feedback Linearization

Having satisfied all mathematical preliminaries, we proceed to calculate the coordinate transformation for system (7) with the choice $h_2 = (\phi_1 + \phi_2)/2$. Based on Eqs. (12), (13), (14), and (15), the coordinate transformation for this system becomes

$$\begin{aligned} \Phi_1 &= h_1 = \theta \\ \Phi_2 &= L_f h_1 = \dot{\theta} \\ \Phi_3 &= L_f^2 h_1 = c_1(\phi_2^2 - \phi_1^2) \\ \Phi_4 &= h_2 = \frac{1}{2}(\phi_1 + \phi_2) \end{aligned} \tag{18}$$

where $\Phi(x) = \mathrm{col}(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ and whose Jacobian is

$$\frac{d\Phi}{dx} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2c_1\phi_1 & 2c_1\phi_2 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Application of the chain rule gives

$$\dot{z} = \left\{ \begin{array}{c} \dot{\theta} \\ c_1\phi_2^2 - c_1\phi_1^2 \\ \frac{2c_1}{N}(\phi_2 v_2 - \phi_1 v_1) + 2c_1c_2\left((g_0 + L_a\theta)\phi_1^2 - (g_0 - L_a\theta)\phi_2^2\right) \\ \frac{1}{2N}(v_1 + v_2) - \frac{c_2}{2}\left((g_0 + L_a\theta)\phi_1 + (g_0 - L_a\theta)\phi_2\right) \end{array} \right\}$$

We place the above result into the form of system (11) where we define β and α as

$$\beta = \frac{1}{N}\begin{bmatrix} -2c_1\phi_1 & 2c_1\phi_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \alpha = \left\{ \begin{array}{c} 2c_1c_2\left((g_0 + L_a\theta)\phi_1^2 - (g_0 - L_a\theta)\phi_2^2\right) \\ -\frac{c_2}{2}\left((g_0 + L_a\theta)\phi_1 + (g_0 - L_a\theta)\phi_2\right) \end{array} \right\} \tag{19}$$

Finally, the nonlinear system in new coordinates (11)is linearized with the feedback law

$$\left\{ \begin{array}{c} v_1 \\ v_2 \end{array} \right\} = \beta^{-1}\left(\left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} - \alpha \right) \tag{20}$$

where $\text{col}(u_1, u_2)$ are the new input into the linearized system and β^{-1} is

$$\beta^{-1} = \frac{N}{(\phi_1 + \phi_2)} \begin{bmatrix} -\frac{1}{2c_1} & 2\phi_2 \\ \frac{1}{2c_1} & 2\phi_1 \end{bmatrix}, \phi_1 + \phi_2 \neq 0$$

As defined, the feedback law of Eq. (20) is regular since both β^{-1} and α are smooth and always finite.

Now consider the diffeomorphism defined with outputs $h_1 = \theta$ and $h_2 = \phi_1\phi_2$. Then the bottom most channel of Eq. (18) becomes $\Phi_4 = \phi_1\phi_2$ so that Jacobian of $\Phi(x)$ is

$$\frac{d\Phi}{dx} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2c_1\phi_1 & 2c_1\phi_2 & 0 & 0 \\ \phi_2 & \phi_1 & 0 & 0 \end{bmatrix}$$

Consequently,

$$\dot{z} = \left\{ \begin{array}{c} \dot{\theta} \\ c_1\phi_2^2 - c_1\phi_1^2 \\ \frac{2c_1}{N}(\phi_2 v_2 - \phi_1 v_1) + 2c_1c_2\left((g_0 + L_a\theta)\phi_1^2 - (g_0 - L_a\theta)\phi_2^2\right) \\ -2c_2 g_0\phi_1\phi_2 + \frac{1}{N}(\phi_2 v_1 + \phi_1 v_2) \end{array} \right\}$$

which when placed in the form of system (11) gives β and α as

$$\beta = \frac{1}{N} \begin{bmatrix} -2c_1\phi_1 & 2c_1\phi_2 \\ \phi_2 & \phi_1 \end{bmatrix} \qquad \alpha = \left\{ \begin{array}{c} 2c_1c_2\left((g_0 + L_a\theta)\phi_1^2 - (g_0 - L_a\theta)\phi_2^2\right) \\ -2c_2 g_0\phi_1\phi_2 \end{array} \right\} \tag{21}$$

Like the previous linearization, the well behaved form of β^{-1} and α implies regular feedback.

$$\beta^{-1} = \frac{N}{(\phi_1^2 + \phi_2^2)} \begin{bmatrix} -\frac{\phi_1}{2c_1} & \phi_2 \\ \frac{\phi_2}{2c_1} & \phi_1 \end{bmatrix}$$

Regardless of the choice of outputs h_2, the feedback linearization results in the linear system of the general form

$$\begin{aligned} \left\{ \begin{array}{c} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{array} \right\} &= \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \\ \begin{array}{ccc} 0 & 0 & 0 \end{array} & [0] \end{bmatrix} \left\{ \begin{array}{c} z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \right\} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} \\ &= Az + Bu \end{aligned} \tag{22}$$

where the linearized system decomposes into two subsystems, the top left of A into a 3×3 system and the bottom right of A into a 1×1 system. For the first subsystem, the equations simplify to

$$\dddot{z}_1 = \dddot{\theta} = u_1$$

while the second subsystem becomes

$$\dot{z}_4 = \dot{\phi}_1 + \dot{\phi}_2 = u_2$$

This implies that the first input can be used to control the beam angle, θ, while the second input can be used to constrain the actuator fluxes independently of one another.

The flux constraint we use has serious repercussions on the total power consumption of the magnetic bearing system. Consider the flux constraint $h_2 = (\phi_1 + \phi_2)/2 = r(t)$, where $r(t)$ is a "dynamic bias" reference signal. The ability to alter the bias flux to achieve better performance is highly desirable for two reasons. First, the bias flux is proportional to bias current, greatly affecting I^2R losses. Second, the force slew capacity of the actuator is directly proportional to the bias flux. Thus, a small bias can be selected during low disturbance periods and a higher bias during greater disturbance periods, thus limiting power losses while maintaining actuator dynamic capacity. Unfortunately, the process of selecting the bias reference trajectory, $r(t)$, doesn't appear to be simple and will require an entirely new layer of control artifice.

On the other hand, the flux constraint $h_2 = \phi_1\phi_2 = \epsilon$, where ϵ is a small constant alluded to in Section 3, discards this problem altogether. With this constraint the flux trajectory behaves parabolically so that both fluxes always have the same sign, while the actuators react nearly complimentarily providing an almost optimal suspension depending on the choice of ϵ. More precisely, the value of ϵ is dictated by dynamic requirements and hardware limitations. For instance the maximum flux density allowed in either actuator is related to magnetic saturation of the iron core of the actuators B_{sat}. On the other hand, the minimum flux allowed must be such that actuator slew capacity is sufficient. Therefore, ϵ must be selected based on hardware limitations. This can be summarized mathematically as

$$\epsilon \geq \phi_{max}\phi_{min} = \left(\frac{\phi_{max}}{A}\right)\left(\frac{NA}{2c_1V_{ps}}\right)\dot{F}_{min} = B_{sat}\dot{F}_{min}\left(\frac{NA}{2c_1V_{ps}}\right) > 0 \tag{23}$$

Here $\dot{F}_{min}$ is the minimum slew rate allowed based on a knowledge of the possible disturbance $d(t)$ (See Eq. (2)), and V_{ps} is the power supply voltage of the amplifiers and represents the maximum voltage available for control. Note, since $\epsilon > 0$, we conveniently avoid the origin (i.e. uncontrollability) automatically.

6 Simulation

The purpose of this simulation is to ascertain the performance of the feedback linearization in a realistic manner. First, we acknowledge that the flux, ϕ, is not directly available for measurement, but instead must be estimated in some way. Therefore it seems necessary to first simulate the system in terms of current and displacement, then to use that information to construct a feedback linearized system. From these "measurements", gap fluxes are estimated and then used to build the linearized feedback of Eqs. (20) and (22). For instance, under ideal circumstances the flux can be derived as

$$\phi_1 = \frac{\mu_0 ANi_1(\phi_1, \theta)}{2(g_0 + L_a\theta)}, \qquad \phi_2 = \frac{\mu_0 AaNi_2(\phi_2, \theta)}{2(g_0 - L_a\theta)} \tag{24}$$

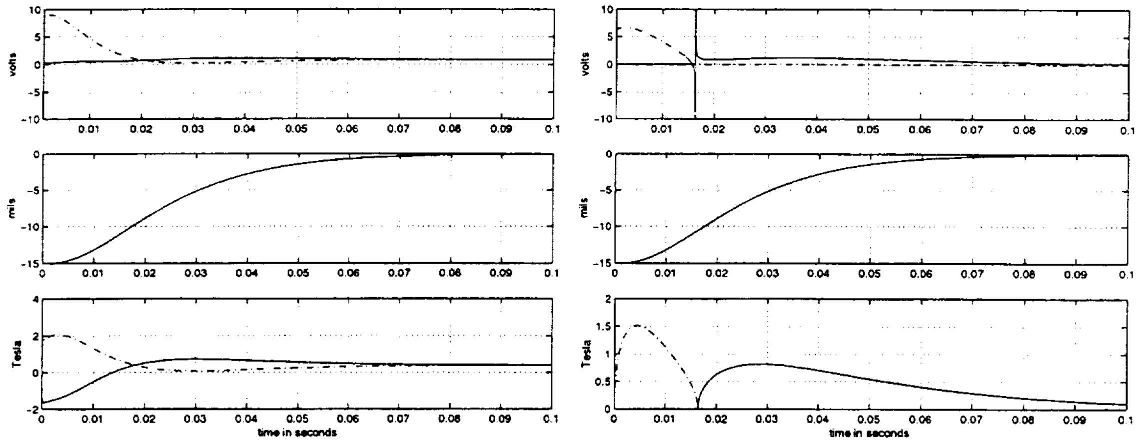

Figure 2: Startup of beam with $h_2 = (\phi_1 + \phi_2)/2$, (left), and $h_2 = \phi_1\phi_2$, (right) with $A1(-)$ and $A2(-\cdot)$

where the coil currents $i_{1,2}$ and beam angle θ are available experimentally. In practice, estimating ϕ is not very simple (Keith [7]).

The system as described has been simulated for both outputs, h_2, described in this paper. In each case, a pole placement controller with identical feedback gains was used to stabilize the linear system. For the linearized system with $h_2 = 1/2(\phi_1 + \phi_2)$, the bias flux was set at a constant of $r(t) = 0.4T$, although, a dynamic bias might just as well have been used. For the linearized system with output $h_2 = \phi_1\phi_2$, ϵ was selected using Eq. (23) such that the steady state flux density is about $0.01T$ ($\epsilon = 10^{-4}$) for each actuator. This choice insures that at *all times* the actuators provide sufficient slew rate to react to any possible disturbances.

Figure (2) shows a startup sequence for the beam system with initial conditions of zero voltage, current, and flux, but leaning against actuator $A1$. The mechanical response of the system was independent of the output employed. Even so, the system with bias required greater startup flux and voltages than the system with constant product. Also, the startup flux for the biased case crosses the 0-axis which **tends** to destabilize the beam since $F \propto \phi^2$. Any dynamic biasing system must avoid this situation. Please note that the flux trajectories for the system with output $h_2 = \phi_1\phi_2$ are at times obscured by the zero axis even though they are slightly positive. Also, note that Eq. (23) guarantees slew performance during these periods. Figure (3) illustrates the system with initial conditions all zero, but with a sinusoidal disturbance torque, $d(t)$, of amplitude 0.8N-m and frequency 2Hz. Again, the mechanical performance in either case was nearly identical. The difference rests with the voltage and flux used to achieve that performance. For instance, the net RMS flux for the biased system was $0.4 \times 2 = 0.8T$ while for the unbiased system the net RMS was about $0.32T$. Lastly, Figure (4) shows the actuators fluxes plotted against one another for both biased and unbiased systems. The fluxes begin at the origin at startup and quickly approach their desired trajectories where they remain indefinitely.

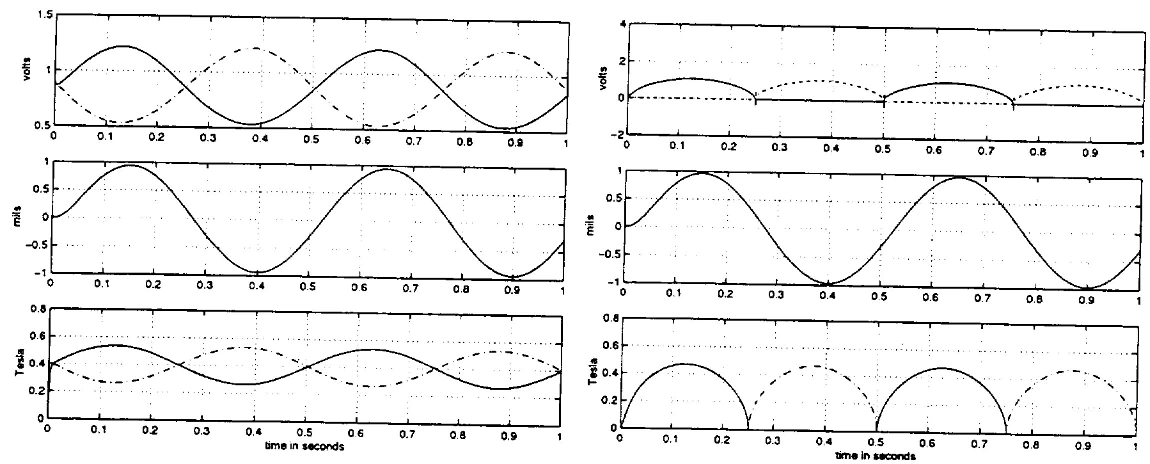

Figure 3: Sinusoidal disturbance for beam with $h_2 = (\phi_1 + \phi_2)/2$, (left), and $h_2 = \phi_1\phi_2$, (right) with $A1(-)$ and $A2(-\cdot)$

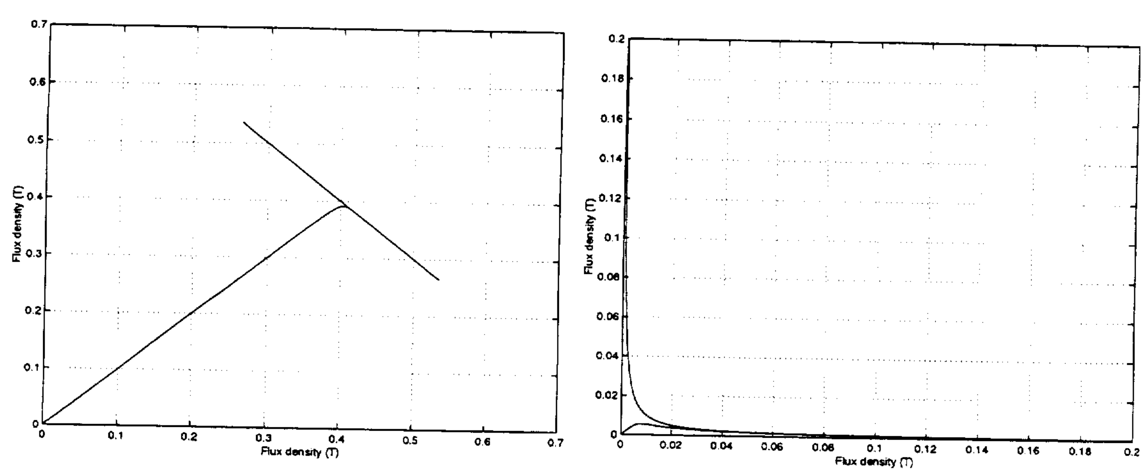

Figure 4: Flux trajectory for beam with $h_2 = (\phi_1 + \phi_2)/2$, (left), and $h_2 = \phi_1\phi_2$, (right) with $A1(-)$ and $A2(-\cdot)$

7 Conclusions

In this paper a systematic and formal mathematical procedure was used to feedback linearize a magnetic bearing system. This procedure is a direct application of the methods illustrated by Isidori [1] and Nijmeijer [8]. Consequently, the resulting linearization had three favorable properties:

(i) linearizing feedback was smooth and finite, i.e. regular;

(ii) the resulting system was controllable; and

(iii) beam angular position and flux regulation achievable.

The generality of the method allows it to be applied to almost any magnetic bearing system. As an additional benefit of the linearization, beam and flux control were decoupled. Specifically, the system decoupled into a third and first order system governing the beam angle and actuator flux respectively (Eq. 22). For the output $h_2 = (\phi_1 + \phi_2)/2$ a dynamic bias was possible. However, there are many difficulties in choosing the trajectory of a dynamic bias: what signals are to be measured to determine reference and do they exist? If these signals are available, how are they processed? how do we optimize the bias trajectory while avoiding slew deprivation? In other words, how do we specify the behavior of the second output h_2? Thus, several barriers must be overcome before dynamic biasing becomes practical. Perhaps, the problem of dynamic biasing is not well posed. Ultimately, our goal is not to achieve dynamic biasing, but instead to achieve some sense of optimization regarding the use of force. If we minimize net force, power losses will necessarily be minimized, motivating the choice of output $h_2 = \phi_1\phi_2$. Constant flux product becomes more appealing since the reference value ϵ can be chosen based on well known actuator properties and the dynamic requirements of the plant.

Future work will consider the application of linear optimal control theory to the linearized system. However, it will be necessary to establish the relation of optimality in the z state and optimality in the x state. If this can be done, then the vast body of literature developed for linear optimal control will be applicable.

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