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Nonlinear Control of Magnetic Bearings

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June 16, 1993

Abstract: *In this paper we present a variety of nonlinear controllers for the magnetic bearing that ensure both stability and robustness. We utilize techniques of discontinuous control to design novel control laws for the magnetic bearing. We present in particular sliding mode controllers, time optimal controllers, winding algorithm based controllers, nested switching controllers, fractional controllers and synchronous switching controllers for the magnetic bearing. We show existence of solutions to systems governed by discontinuous control laws, and prove stability and robustness of the chosen control laws in a rigorous setting. We design sliding mode observers for the magnetic bearing, and prove the convergence of the state estimates to their true values. We present simulation results of the performance of the magnetic bearing subject to the aforementioned control laws, and conclude with comments on design.*

1   Introduction

Magnetic bearings require high precision control systems to ensure adequate stability, stiffness and robustness. Magnetic bearing controllers are typically linear controllers that perform pole placement through state feedback. Nonlinear control techniques on the other hand provide surprisingly simple, yet stable and robust controllers. It is our contention that nonlinearities can be introduced and exploited in a controlled manner to provide significant performance enhancements without increasing the complexity of control. To this end we present the following control techniques and controllers for the magnetic bearing.

- Sliding mode control of magnetic bearings.
- Minimum time control of magnetic bearings.
- Winding algorithm based control of magnetic bearings.
- Nested switching control of magnetic bearings.
- Synchronous control of magnetic bearings.
- Fractional control of magnetic bearings.
- Sliding mode observers for magnetic bearings.

Each of the aforementioned control methodologies utilizes discontinuous control of the magnetic bearing. We show rigorously the existence of solutions to these highly nonlinear systems and prove the stability and convergence of the magnetic bearing system subject to each of these control laws.

The organization of this paper is as follows. In the first section we present the basics of the theory of differential equations with discontinuous righthand sides. The second section presents the dynamical equations of the magnetic bearing. The third section presents the theory, proof and simulation of a sliding mode controller of a magnetic bearing. In the fourth section we present the theory, proof and simulation of a minimum time controller of a magnetic bearing. Section V of this paper presents winding algorithm based control of the magnetic bearing. The sixth section presents the theory, proof and simulation of a nested switching controller of a magnetic bearing and is followed by the synchronous controller in the eighth section. The ninth section of this paper presents a fractional controller and a conjecture pertaining to the stability of the fractional controller. We conclude this paper with a presentation of sliding mode observers in the tenth section. The appendix lists facts and definitions from real analysis useful towards understanding the mathematics presented in this paper.

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2 Differential Equations with Discontinuous Righthand Sides

As a prelude we compare and classify ordinary differential equations based on the nature of their right hand sides. Consider a differential equation of the following form.

$$\dot{x} = f(x, t) \tag{1}$$
$$x(0) = x_0 \tag{2}$$
$$x \in \Re^n \ t \in \Re_+ \tag{3}$$
$$f(x, t) : \Re^n \times \Re_+ \to \Re^n \tag{4}$$
$$\tag{5}$$

The smoothness assumptions on $f(x, t)$ determine the kind of differential system referred to by (1).
The three major kinds of differential systems are

1. **Cauchy Differential Systems:** In the domain D of the (x, t) space,

   - $f(x, t) : \Re^n \times \Re_+ \to \Re^n$ is continuous in x.
   - $f(x, t) : \Re^n \times \Re_+ \to \Re^n$ is continuous in t.

2. **Caratheodory Differential Systems:** In the domain D of the (x, t) space,

   - $f(x, t) : \Re^n \times \Re_+ \to \Re^n$ is continuous in x.
   - $f(x, t) : \Re^n \times \Re_+ \to \Re^n$ is discontinuous in t on sets of zero measure.

3. **Filippov Differential Systems:** In the domain D of the (x, t) space,

   - $f(x, t) : \Re^n \times \Re_+ \to \Re^n$ is discontinuous in x and t on sets of zero measure.

The nature of the righthand sides indicates the kinds of solutions (strong or weak) that exist for the differential system. From a control systems engineering standpoint, the use of discontinuous controls necessitates the need for a careful stability analysis owing to the nature of the solutions that exist for such systems.

2.1 Filippov Differential Systems

In this section we will develop solution concepts and conditions for existence of solutions to differential equations with discontinuous right hand sides. Such equations represent physical systems governed by switching behaviours.

Instead of describing solutions for differential equations with discontinuous right hand sides, we will consider differential inclusions which include the said discontinuity as a special case. We will then describe generalized solution concepts for these differential inclusions, and will present conditions for existense of generalized solutions to differential inclusions.

We consider Filippov Differential Systems of the following form.

$$\dot{x} = f(x, t) \tag{6}$$
$$x(t = 0) = x_0 \tag{7}$$
$$x \in \Re^n \ t \in \Re_+ \tag{8}$$
$$f(x, t) : \Re^n \times \Re_+ \to \Re^n \tag{9}$$
$$\tag{10}$$

where in the domain D of the (x, t) space,

- $f(x, t) : \Re^n \times \Re_+ \to \Re^n$ is discontinuous in $x \in D$ on sets of zero measure.
- $f(x, t) : \Re^n \times \Re_+ \to \Re^n$ is discontinuous in $t \in D$ on sets of zero measure.
- $f(x, t) : \Re^n \times \Re_+ \to \Re^n$ is measurable in $t \in D$ for each $x \in D$.
- $\|f(x, t)\| \leq K_f(t) \ \forall \ (x, t) \in D$ where $K_f(t) : \Re_+ \to \Re$ is summable.

The aforementioned conditions on the function $f(x, t) \Re^n \times \Re_+ \to \Re^n$ are also called Filippov conditions.

We will now consider a differential inclusion that adequately describes the discontinuous system. Though the function $f(x, t) : \Re^n \times \Re_+ \to \Re^n$ of equation (6) is undefined on sets of zero measure, we choose instead to represent the function $f(x, t) : \Re^n \times \Re_+ \to \Re^n$ by a set valued map on such sets of zero measure. That is to say, if for instance the function is undefined at a point $(x^*, t^*) \in \Re^n \times \Re_+$, we formally define the function to be set valued at the point (x^*, t^*). Indeed, depending on the set-value attributed to the function at the point (x^*, t^*), we may show the existence of certain generalized solutions to the system (6). To construct the inclusion intelligently, we need some knowledge about the behaviour of the function $f(x, t) : \Re^n \times \Re_+ \to \Re^n$, in a neighbourhood of the point of

discontinuity. To justify the use of the inclusion, we must show that given any arbitrary $\epsilon \in \Re_+$, there exists a small enough $\delta \in \Re_+$ neighbourhood of the point of discontinuity, such that, the trajectories of the differential equation in this δ neighbourhood are ϵ close to the solutions of the differential inclusion. Furthermore, as the size of the set containing the point of discontinuity shrinks to zero, that is $\delta \to 0$, the solutions of the differential equation tend to the solution of the differential inclusion. That is to say, that the trajectories of the differential equation weakly converge to the solution of the differential inclusion. We will say more about this later.

Indeed, given a discontinuous differential system of the form (6), henceforth we will replace it (whenever possible) with a *differential inclusion* of the following form.

$$\dot{x} \quad \in \quad \mathcal{F}(x,t) \tag{11}$$

$$x(t=0) \quad = \quad x_0 \tag{12}$$

$$x \in \Re^n \ t \in \Re_+ \tag{13}$$

$$\mathcal{F}(x,t) : \Re^n \times \Re_+ \to S \in \Re^n \tag{14}$$

$$\tag{15}$$

where S is a set in $\Re^n$ and in the domain D of the (x,t) space,

- the set valued map $\mathcal{F}(x,t) : \Re^n \times \Re_+ \to S \in \Re^n$ is *upper semi-continuous*.
- The $Range[\mathcal{F}(x,t)] \in \Re^n$ is *compact and convex*.

Comment 2.1 *The definition of the inclusion $\mathcal{F}(x,t)$ is such that it is single-valued in the domain of continuity of the function $f(x,t)$; indeed it is equal to $f(x,t)$ in the domains of continuity, but is set valued in the domains of discontinuity of $f(x,t)$.*

Comment 2.2 *It is important to note the properties of the set $S \in \Re^n$ which will be used for the existence of solutions.*

We now formally define the solution of a Filippov differential system.

Filippov Solution Concept: *An absolutely continuous vector function $s(t) : \Re_+ \to \Re^n$ is defined to be a Filippov solution of the Filippov differential system (11) if for almost all $t \in D$,*

$$\frac{ds}{dt}\Big|_{t=t^*} \in \mathcal{F}(s(t^*), t^*) \tag{16}$$

where

$$\mathcal{F}(s(t^*), t^*) \quad = \quad f(s(t^*), t^*) \ \text{in the domains of continuity} \tag{17}$$

$$\mathcal{F}(s(t^*), t^*) \quad = \quad \bigcap_{\delta > 0} \bigcap_{\mu N = 0} convex\text{-}hull(B(x,\delta) - N, t) \tag{18}$$

and $\bigcap_{\mu N=0}$ denotes the intersection over all sets N of Lebesgue measure zero where the function $f(x,t)$ is either undefined or discontinuous.

Comment 2.3 *In the domains of continuity of $f(x,t) : \Re^n_+ \to \Re^n$, the inclusion $\mathcal{F}(x,t)$ is the same as the function and therefore the set operation $\in$ in equation (158) must be replaced with the strict equality $=$*

The utility of the Filippov solution concept is that it is indeed the limit of solutions to (6) averaged over neighbourhoods of diminishing size. The key point to be understood is that the Filippov trajectories of the discontinuous system remain close to the true trajectories.

As is evidenced in the proofs of the Cauchy and Caratheodory systems, the method of constructing solutions to differential equations begins by constructing sequences of approximating solutions, and then ensuring that the approximations converge in some sense.

We now state the theorem that guarantees the local existence of Filippov solutions.

Theorem 2.1 *Local Existence Of Filippov Solutions To Filippov Differential Systems*

Given *(G1)* *A Filippov differential system of the form (11).*

If *(I1)* *The domain D where $f(x,t)$ is specified for almost all t*
$(x,t) \in \Re^n \times \Re_+ : ||x - x_0|| \le K_x$ *and* $t \le K_t$

(I2) *$f(x,t) : \Re^n \times \Re_+ \to \Re^n$ is measurable in $t \in \Re_+$ for all $x \in \Re^n$*

(I3) *$||f(x,t)|| \le K_f(t) \ \forall \ (x,t) \in D$ where $K_f(t) : \Re_+ \to \Re$ is summable. Furthermore there exists $K_f \in \Re_+$ such that $K_f > |K_f(t)| \ \forall \ t \in D$*

(I4) *The differential inclusion in (11), $\mathcal{F}(x,t) : \Re^n \times \Re_+ \to S \in \Re^n$, where S is a set in $\Re^n$ and in the domain D of the (x,t) space satisfies the following two assumptions.*

- *the set valued map $\mathcal{F}(x,t) : \Re^n \times \Re_+ \to S \in \Re^n$ is upper semi-continuous.*
- *the set $S \in \Re^n$ is compact and convex.*

Then *(T1)* *The differential system (11) has at least one Filippov solution $s(t) : \Re_+ \to \Re^n$ for $t \le min(K_t, \frac{K_x}{K_f})$ satisfying the initial condition $s(0) = x_0$.*

3 Dynamic Equations of the Magnetic Bearing

The dynamic equations of the magnetic bearing [RG93] may be written as follows.

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0_{4\times4} & I_{4\times4} \\ 0_{4\times4} & A[\omega] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u
\tag{19}
$$

$$
y = \begin{bmatrix} I_{4\times4} & 0_{4\times4} \end{bmatrix} \begin{bmatrix} x_1^i \\ x_2^i \end{bmatrix}
\tag{20}
$$

where,

$$
x_1^i \in \Re^4 = \begin{bmatrix} x_1^1 \\ x_1^2 \\ x_1^3 \\ x_1^4 \end{bmatrix}
\tag{21}
$$

$$
x_2^i \in \Re^4 = \begin{bmatrix} x_2^1 \\ x_2^2 \\ x_2^3 \\ x_2^4 \end{bmatrix}
\tag{22}
$$

$$
u \in \Re^4 = \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}
\tag{23}
$$

$$
y \in \Re^4 = \begin{bmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{bmatrix}
\tag{24}
$$

$$
A : \Re^+ \to \Re^{4\times4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a\omega \\ 0 & 0 & a\omega & 0 \end{bmatrix}
\tag{25}
$$

$$
I \in \Re^{4\times4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\tag{26}
$$

$$
B \in \Re^{4\times4} = \begin{bmatrix} \frac{1}{m} & \frac{1}{m} & 0 & 0 \\ 0 & 0 & \frac{1}{m} & \frac{1}{m} \\ 0 & 0 & -l_1 & l_2 \\ l_1 & -l_2 & 0 & 0 \end{bmatrix}
\tag{27}
$$

$$
\tag{28}
$$

The state variables $x_1 \in \Re^4$ are the generalized positions and rotational angles while the state variables $x_2 \in \Re^4$ are the generalized linear and angular velocities.

Now choose a control law of the following form.

$$
u = B^{-1}[-A[\omega]x_2 + v]
\tag{29}
$$

$$
v = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{bmatrix}
\tag{30}
$$

where the control inputs $v \in \Re^4$ will be specified later. Substituting control law (29) in the dynamical equations (19) - (20), we arrive at the decoupled form of the state equations written as follows.

$$
\dot{x}_1^i = x_2^i
\tag{31}
$$

$$
\dot{x}_2^i = v^i \quad i = 1, 2, \ldots, 4
\tag{32}
$$

4 Sliding Mode Control of the Magnetic Bearing

In this section, we specialize the theory of discontinuous systems to a special class of systems of the following form.

$$\dot{x} = f_+(x) \text{ for } [x : s(x) > 0] \qquad (33)$$

$$= f_-(x) \text{ for } [x : s(x) < 0] \qquad (34)$$

where $x \in \Re^n$, and $f(x) : \Re^n \to \Re^n$ and $s(x) : \Re^n \to \Re$. Note that $S = \{x : s(x) = 0\}$ is a manifold of dimension $n - 1$. This manifold S is called the *sliding manifold* or *sliding surface*. The dynamics of the system on this manifold S is called the *sliding dynamics* or *sliding modes* of the system. The design of the manifold S is such that it is globally attractive, and trajectories commencing from arbitrary initial conditions reach S in finite time. Furthermore, the dynamics on S achieves the control objective.

Local existence of solutions is verified by modelling the system represented by equations (33) - (34) by the appropriate differential inclusions and verifying whether the inclusion satisfies the hypotheses of the theorem concerning local existence of Filippov solutions.

Uniqueness, in the sense of the Filippov solution is shown if either $\frac{\partial s(x)}{\partial x} f_+(x) < 0$ or $\frac{\partial s(x)}{\partial x} f_-(x) > 0$. This is shown in [SS83], [Fil88], [Fil61]. The physical interpretation of these conditions is simply that the trajectories of the system are always directed towards S, thus rendering it attractive.

Example 4.1

$$\dot{x} = -k sgn[x] \qquad (35)$$

$$sgn[x] = 1 \text{ if } x > 0 \qquad (36)$$

$$sgn[x] = -1 \text{ if } x < 0 \qquad (37)$$

Modelling the system (35) by a simple differential inclusion, we rewrite (35) as

$$\dot{x} \in \mathcal{F}(x) \qquad (38)$$

where

$$\mathcal{F}(x) = sgn[x] \text{ if } x \neq 0 \qquad (39)$$

$$\mathcal{F}(x) \in [-1,1] \text{ if } x = 0 \qquad (40)$$

The inclusion in (38) is closed, bounded, convex and uppersemicontinuous and therefore by the theorem on existence of Filippov solutions, Filippov solutions exist for this system.

The sliding modes of a system, defined to be the Filippov solutions to the system on the manifold S, are calculated by performing Filippov averaging, which is a convex combination of dynamics on either side of the manifold S. Indeed, by dynamics on either side of the manifold S, we merely refer to $f_+(x)$ and $f_-(x)$. The simple extension of the notion of sliding manifolds to non-autonomous systems is shown in [SS83].

While the theory of existense of solutions has been developed for general nonlinear systems with discontinuous controls, the methodology to design sliding mode controls to achieve stabilization or tracking is well understood only for a restricted class of systems [SRS91]. In the following sections, we will present the theory for Linear Time Invariant Systems - SISO and MIMO.

4.1 Sliding Mode Design For LTI Systems

Consider linear time invariant systems represented by the following equations

$$\dot{\bar{x}} = A\bar{x} + Bu \qquad (41)$$

where $\bar{x} \in \Re^n$, $A \in \Re^{n \times n}$, $B \in \Re^{m \times n}$ and the controls $u \in \Re^m$. We will now prescribe the sliding mode controller design procedure in a sequence of steps.

Step 1.

Check to see if the system is completely controllable. If the system is not completely controllable, a sliding mode controller cannot be designed.

Step 2.

If the system is completely controllable, find a linear transformation of the state that recasts the system in the controllable canonical form. That is find a transformation

$$x = T\bar{x} \quad T \in \Re^{n \times n} \qquad (42)$$

such that the state equations are of the form

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \cdots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \\ b_1 & b_2 & \cdots & b_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \qquad (43)$$

Step 3.

We define $S(x) : \Re^n \to \Re$ as

$$S(x) = a_1 x_1 + a_2 x_2 + \cdots + a_{n-1} x_{n-1} + x_n \tag{44}$$

where the coefficients a_i $i = 1, 2, \ldots, n - 1$ of (44) are such that the polynomial $S(x)$ is a Hurwitz polynomial. Furthermore, note that $S = 0$ is an $n - 1$ dimensional manifold, called the sliding surface.

Indeed now choose the control input u to be

$$u(t) = -b_1 x_1 - b_2 x_2 \cdots - b_n x_n - v_1(t) \tag{45}$$

$$v(t) = -a_1 x_2 - a_2 x_3 - \cdots - a_{n-1} x_n - k sgn[S(x)] \tag{46}$$

Choice of control u enables us to rewrite the system in the form

$$\dot{x}_1 = x_2 \tag{47}$$

$$\dot{x}_2 = x_3 \tag{48}$$

$$\dot{x}_{n-1} = -a_1 x_1 - a_2 x_2 - \cdots - a_{n-1} x_{n-1} + S(x) \tag{49}$$

$$\dot{S}(x) = -k sgn[s(x)] \tag{50}$$

It is easy to show that Filippov solutions exist, and that $S(x) = 0$ is reached in finite time from arbitrary initial conditions. Furthermore on the $n - 1$ dimensional manifold $S = 0$, the reduced order dynamics is exponentially stable. Consequently global exponential stability of the system is shown.

The choice of discontinuous input induces chatter in the system. To reduce the chatter, we utilize various regularizations and smoothings of the discontinuous sgn function. The common smoothing technique is the use of the saturation function, which is presented in [SS83].

We now present a choice of *continuous control input* that enables us to reach the sliding surface $S = 0$ in finite time. Indeed, consider the control given by

$$u(t) = -b_1 x_1 - b_2 x_2 \cdots - b_n x_n - v_1(t) \tag{51}$$

$$v(t) = -a_1 x_2 - a_2 x_3 - \cdots - a_{n-1} x_n - k|S(x)|^{\frac{1}{m}} sgn[S(x)] \tag{52}$$

$$m > 1 \tag{53}$$

Such a choice of control u enables us to recast the system equations in the form

$$\dot{x}_1 = x_2 \tag{54}$$

$$\dot{x}_2 = x_3 \tag{55}$$

$$\dot{x}_{n-1} = -a_1 x_1 - a_2 x_2 - \cdots - a_{n-1} x_{n-1} + S(x) \tag{56}$$

$$\dot{S}(x) = -k|S(x)|^{\frac{1}{m}} sgn[s(x)] \tag{57}$$

It is easy to show that Filippov solutions exist, and that the $n - 1$ dimensional manifold $S(x) = 0$ is reached in finite time. Furthermore on the $n - 1$ dimensional manifold given by $S = 0$, we see that the reduced order dynamics is exponentially stable. Consequently global exponential stability of the system is shown. This control law u is interesting in that it is continuous, but not differentiable.

Comment 4.1 The disturbance rejection properties of the discontinuous control law are significantly better than that of the continuous control law. This indeed is the design tradeoff involved in designing continuous control laws.

Comment 4.2 The extension of the sliding mode control techniques to controllable MIMO systems that are decouplable is trivial. Once the system equations are transformed into decoupled systems, each of which is in the controllable canonical form, we apply the design method outlined earlier to design sliding surfaces for the decoupled system. Note however that sliding occurs not at the individual surfaces, but at the intersection of all these surfaces.

Theorem 4.1 *Sliding mode control of a magnetic bearing system.*

Given *(G1)* A magnetic bearing system of the form *(31) - (32)*.

If *(I1)* The controls v^i $i = 1, 2, \ldots, 4$ are chosen as

$$v^i_{sliding} = -a^i_1 x^i_2 - k * sgn[a^i_1 x^i_1 + x^i_2] \tag{58}$$

Then *(T1)* Filippov solutions exist for the system *(31) - (32)* subject to the control law *(58)*.

(T2) The trajectories of the system *(31) - (32)* subject to the control law *(58)* reach the origin in finite time.
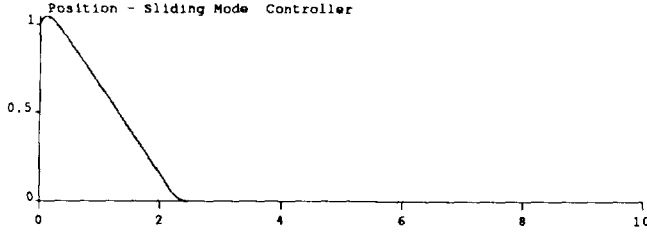
Figure 1: Sliding Mode Control of a Single Axis of a Magnetic Bearing

Proof: ♠ ▷ To show the existence of generalized Filippov solutions we first note that the dynamical system represented by equations (31) - (32) subject to the control law (58) can be modelled by the following inclusion.

$$\dot{x}_1^i = -a_1^i x_1^i + S \tag{59}$$

$$\dot{S} \in \mathcal{F}^i(x) \tag{60}$$

where the inclusions $\mathcal{F}^i(x) : \Re \to [-k, k]$ are specified as

$$\mathcal{F}^i(x) = -k * sgn[a_1^i x_1^i + x_2^i] \text{ if } ||x||_2 > 0 \tag{61}$$

$$\in [-k, k] \text{ else} \tag{62}$$

$$i = 1, 2, \ldots, 4 \tag{63}$$

The inclusions $\mathcal{F}^i(x)$ $i = 1, 2, \ldots, 4$ are

- closed, bounded, convex and uppersemicontinuous.

Invoking the theorem on the existence of generalized Filippov solutions, we conclude that Filippov solutions exist for the system (31) - (32) subject to the control law (58).

Stability and robustness of the magnetic bearing follow from the earlier discussions. ◁ ♠

The phase portrait of trajectories subject to the sliding mode control is given below.

5 Minimum Time Control of the Magnetic Bearing

It is many times desirable in a magnetic bearing to choose a control law to perform regulation in minimum time. Such minimum time regulation ensures good response to impulsive perturbation forces. To achieve regulation in minimum time, we formulate the optimal control problem as specified in [AEB75].

Consider the minimum time optimal control problem with the functional to be minimized, given by

$$J = \int_0^{t_f} dt \tag{64}$$

Theorem 5.1 *Minimum time control of a magnetic bearing system.*

Given *(G1) A magnetic bearing system of the form (31) - (32).*

(G2) A functional to be extremized of the form (64).

If *(I1) The controls v^i $i = 1, 2, \ldots, 4$ are chosen as*

$$v_{optimal}^i = \begin{cases} -sgn[x_1^i + \frac{x_2^i |x_2^i|}{2}] & if \ |x_1^i + \frac{x_2^i |x_2^i|}{2}| > 0 \\ -sgn[x_2^i] & if \ |x_1^i + \frac{x_2^i |x_2^i|}{2}| = 0 \end{cases} \tag{65}$$

33

Then *(T1)* *The trajectories of the system (31) - (32) subject to the control law (65) reach the origin in minimum time.*

Proof: ◆ ▷ Using standard methods of optimal control [AEB75], we write down the Hamiltonian function $H(x, u, \lambda, t)$ as

$$H(x, u, \lambda, t) = 1 + \lambda_1 x_2^i + \lambda_2 v^i \tag{66}$$

Inspection of equation (66) reveals that the control v^i that minimizes the Hamiltonian is given by

$$v^i = -sgn[\lambda_2] v_{max}^i \tag{67}$$

where v_{max}^i is the maximum permissible value of control. Without loss of generality, we will assume that $v_{max}^i = 1$. where λ_1 and λ_2 are the co-state variables. The co-state equations are given by

$$\dot{\lambda}_1 = 0 \tag{68}$$

$$\dot{\lambda}_2 = -\lambda_1 \tag{69}$$

Integrating the co-state equations yields

$$\lambda_2(t) = -\lambda_1(0)t - \lambda_2(0) \tag{70}$$

Therefore the optimal control is given as

$$v^i = sgn[-\lambda_1(0)t - \lambda_2(0)] \tag{71}$$

The control can assume only two values $+1$ or -1. When $v^i = +1$, we integrate the state equations to obtain

$$x_2^i(t) = t + x_2^i(0) \tag{72}$$

$$x_1^i(t) = \frac{t^2}{2} + x_2^i(0)t + x_1^i(0) \tag{73}$$

Eliminating t we obtain $\tag{74}$

$$x_1^i = \frac{[x_2^i]^2}{2} + x_1^i(0) - \frac{[x_2^i]^2(0)}{2} \tag{75}$$

Similarly, when $v^i = -1$, integrating the state equations we obtain

$$x_2^i(t) = -t + x_2^i(0) \tag{76}$$

$$x_1^i(t) = -\frac{t^2}{2} + x_2^i(0)t + x_1^i(0) \tag{77}$$

Eliminating t we obtain $\tag{78}$

$$x_1^i = -\frac{[x_2^i]^2}{2} + x_1^i(0) - \frac{[x_2^i]^2(0)}{2} \tag{79}$$

These curves describe a family of parabolas, whose switching curve may be written as

$$S(x, t) = x_1^i + \frac{x_2^i |x_2^i|}{2} \tag{80}$$

In terms of the switching curve, the control $v_{optimal}^i$ may be written as

$$v_{optimal}^i = \begin{cases} -sgn[x_1^i + \frac{x_2^i|x_2^i|}{2}] & \text{if } |x_1^i + \frac{x_2^i|x_2^i|}{2}| > 0 \\ -sgn[x_2^i] & \text{if } |x_1^i + \frac{x_2^i|x_2^i|}{2}| = 0 \end{cases} \tag{81}$$

◁ ◆

The phase portrait of trajectories subject to the optimal control $v_{optimal}^i$ is given below. Note the trajectories converging to the switching curve, which is nonlinear (while the switching curve in conventional sliding mode systems is linear). The chosen control gains are
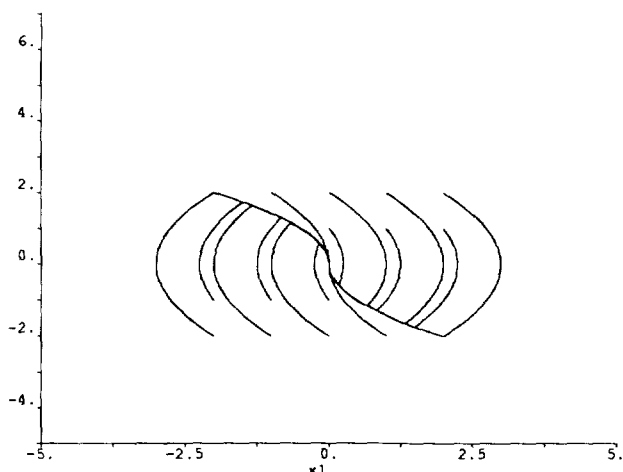
$$k_1 = 1 \tag{82}$$

$$k_2 = 2 \tag{83}$$

Figure 2: Minimum Time Control of a Single Axis of a Magnetic Bearing

6 Winding Algorithm Based Control of the Magnetic Bearing

The winding algorithm was introduced by [Kor68], [Pra92] and makes use of continuous switching between the surfaces $x_1^i = 0$ and $x_2^i = 0$ to reach the origin. The interesting feature of this control technique is that the control has two switches. One switch is used to change the direction, and the other is used to change the magnitude. By repeatedly switching between the surfaces $x_1^i = 0$ and $x_2^i = 0$, we wind closer to the origin.

Theorem 6.1 *Magnetic bearing control utilizing the winding algorithm.*

Given *(G1)* *A magnetic bearing system of the form (31) - (32).*

If *(I1)* *The controls* v^i $i = 1, 2, \ldots, 4$ *are chosen as*

$$v_{winding}^i = -k_1 sgn[x_1^i] - k_2 sgn[x_2^i] \quad k_1 > k_2 > 0 \tag{84}$$

Then *(T1)* *Filippov solutions exist for the system (31) - (32) subject to the control law (84).*

(T2) *The trajectories of the system (31) - (32) subject to the control law (84) wind to the origin in finite time.*

Proof: ♠ ▷ To show the existence of generalized Filippov solutions we first note that the dynamical system represented by equations (31) - (32) subject to the control law (84) can be modelled by the following inclusion.

$$\dot{x}_1^i = x_2^i \tag{85}$$
$$\dot{x}_2^i \in \mathcal{F}^i(x) \tag{86}$$

where the inclusions $\mathcal{F}^i(x) : \Re \to [-(k_1 + k_2), (k_1 + k_2)]$ are specified as

$$\mathcal{F}^i(x) = -k_1 sgn[x_1^i] - k_2 sgn[x_2^i] \text{ if } \|x\|_2 > 0 \tag{87}$$
$$\in [-(k_1 + k_2), (k_1 + k_2)] \text{ else} \tag{88}$$
$$i = 1, 2, \ldots, 4 \tag{89}$$

The inclusions $\mathcal{F}^i(x)$ $i = 1, 2, \ldots, 4$ are

- closed, bounded, convex and uppersemicontinuous.

Invoking the theorem on the existence of generalized Filippov solutions, we conclude that Filippov solutions exist for the system (31) - (32) subject to the control law (84).

Let us first prove the stability and finite time stabilization of the algorithm. To show stability, we use the extended Lyapunov theorem, [AKP91] proofs for which may be found in [AC84]. The theorem is primarily used to conclude *weak-stability* of differential inclusions by investigating generalized gradients of non-differentiable Lyapunov functions. A brief statement of the theorem would be as follows.

Given a differential inclusion $\dot{x} \in F(x, t)$ *and a nondifferentiable Lyapunov function* $V(x)$. *If for every element* v *in the generalized gradient of* V, *there exists at least one element* $f \in F(x, t)$, *such that* $L_F V \leq 0$, *then the zero-solution is weakly asymptotically stable.* Indeed, weak asymptotic stability is the best we could hope for when dealing with set-valued differential inclusions.

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Now consider the system (31) - (32) subject to the controls $u_{winding}$. The system equations are

$$\dot{x}_1 = x_2^i \tag{90}$$

$$\dot{x}_2 = -k_1 sgn[x_1^i] - k_2 sgn[x_2^i] \tag{91}$$

Consider a candidate Lyapunov function

$$V = |x_1^i| + \frac{x_2^i}{2k_1} \tag{92}$$

The derivative for $x_1^i, x_2^i \neq 0$ is given by

$$\dot{V} = -\frac{k_2 x_2^2}{k_1} \tag{93}$$

$$\leq 0 \tag{94}$$

Therefore $x_2^i \to 0$, and the reduced dynamics is such that $x_1^i \to 0$. However, when $x_1^i = 0$, it is clear that we have to investigate the properties of the generalized gradient of V. However, it is obvious that when $x_1^i = 0$, for every element v of the generalized gradient of V, (which in this case happens to be any real number in (-1,1)) there exists an element of the inclusion $F(x,t)$ (indeed, choose $f = v$) such that the generalized gradient of V along the flow of the inclusion $F(x,t)$ is negative definite. The conditions of the generalized Lyapunov theorem are satisfied, and hence the result.

Finite time is shown by considering the state equations of the planar dynamical system in the various quadrants. Indeed, if the portrait of the system were to be drawn with x_1^i along the x axis and x_2^i along the y axis, we would note the following.

$$x_1^i = \pm \frac{[x_2^i]^2}{k_1 + k_2} \text{ in the first and third quadrants} \tag{95}$$

$$x_2^i = \mp \frac{[x_2^i]^2}{k_1 - k_2} \text{ in the second and fourth quadrants} \tag{96}$$

Every instance the trajectory moves from the first quadrant through the fourth quadrant to hit the y axis, we see a contraction occurring in the magnitude of x_2^i in the following manner.

$$[x_2^i]^2(t_1) = \frac{k_1 - k_2}{k_1 + k_2}[x_2^i]^2(0) \tag{97}$$

From the third quadrant through the second to strike the y axis again, we see the following contraction.

$$[x_2^i]^2(t_2) = \frac{k_1 - k_2}{k_1 + k_2}[x_2^i]^2(t_1) \tag{98}$$

The state trajectory therefore winds to the origin. ◁♠

The phase portrait of the planar dynamical system subject to the winding algorithm is illustrated below. Note the very interesting way in which the state trajectories wind to the origin. The values of chosen control gains are

$$k_1 = 2 \tag{99}$$

$$k_2 = 1 \tag{100}$$

7 Nested Switching Control of the Magnetic Bearing

Nested switching controls work well for planar dynamical systems [Pra92]. The basic approach is to permit chatter about the dual sliding surfaces $x_1^i = 0$ and $x_2^i = 0$. It is to be noted that chatter for multiple sliding surfaces is the equivalent of limit-cycle like behaviour. Consequently, by utilizing multiple sliding surfaces, and nondifferentiable controls, we are willing to tolerate limit-cycle like behaviour at the origin. Indeed, the problems associated with eliminating chatter in one-dimensional systems naturally extend to the higher order systems also. The use of saturation functions to perform nested switching is an extension of the idea of using saturation functions in one-dimensional systems, to many dimensions.

Theorem 7.1 *Magnetic bearing control utilizing the winding algorithm.*

Given *(G1) A magnetic bearing system of the form (31) - (32).*

If *(I1) The controls v^i $i = 1, 2, \ldots, 4$ are chosen as*

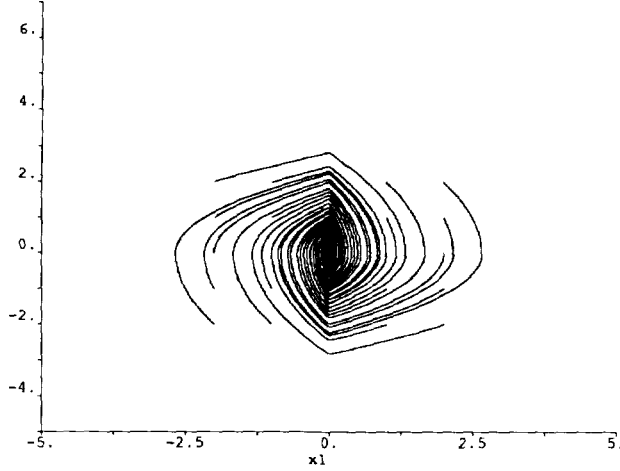$$v_{nested}^i = -k_2 sgn[x_2^i - k_1 sgn[x_1^i]] \tag{101}$$

Figure 3: Winding Algorithm Based Control of a Single Axis of a Magnetic Bearing

Then *(T1) Filippov solutions exist for the system (31) - (32) subject to the control law (101).*

(T2) The trajectories of the system (31) - (32) subject to the control law (101) reach the origin in finite time.

Proof: ♠ ▷

To show the existence of generalized Filippov solutions we first note that the dynamical system represented by equations (31) - (32) subject to the control law (101) can be modelled by the following inclusion.

$$\dot{x}_1^i = x_2^i \tag{102}$$

$$\dot{x}_2^i \in \mathcal{F}^i(x) \tag{103}$$

where the inclusions $\mathcal{F}^i(x): \Re \to [-(k_1 + k_2), (k_1 + k_2)]$ are specified as

$$\mathcal{F}^i(x) = -k_2 sgn[x_2^i - k_1 sgn[x_1^i]] \text{ if } ||x||_2 > 0 \tag{104}$$

$$\in [-k_2, k_2] \text{ else} \tag{105}$$

$$i = 1, 2, \ldots, 4 \tag{106}$$

The inclusions $\mathcal{F}^i(x)$ $i = 1, 2, \ldots, 4$ are

- closed, bounded, convex and uppersemicontinuous.

Invoking the theorem on the existence of generalized Filippov solutions, we conclude that Filippov solutions exist for the system (31) - (32) subject to the control law (84).

Consider the system (31)-(32) subject to the nested switching control law given by

$$\dot{x}_1 = x_2^i \tag{107}$$

$$\dot{x}_2 = -k_2 sgn[x_2^i + k_1 sgn[x_1^i]] \tag{108}$$

Now consider the following nondifferentiable Lyapunov function

$$V = \frac{[x_2^i + k_1 sgn[x_1^i]]^2}{2} \tag{109}$$

$$\dot{V} = [x_2^i + k_1 sgn[x_1^i]][\dot{x}_2 + 0] \text{if } |x_1^i| > 0 \tag{110}$$

$$= -k_2 |x_2^i + k_1 sgn[x_1^i]| \tag{111}$$

$$\leq 0 \tag{112}$$

Therefore $x_2^i \to -k_1 sgn[x_1^i]$. Indeed, it is easy to see that this happens in finite time. As in finite time $x_2^i = -k_1 sgn[x_1^i]$; now consider the Lyapunov function
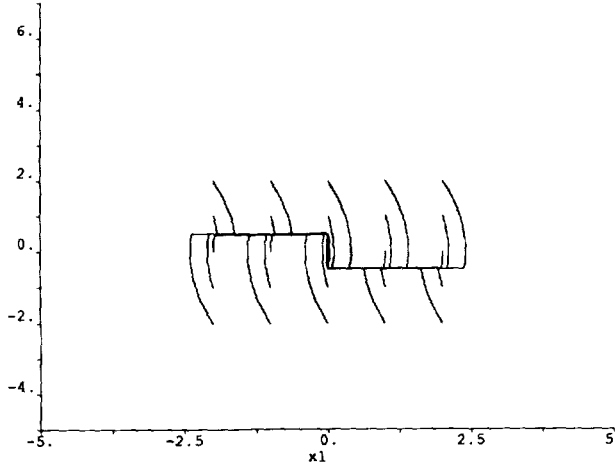
$$V_1 = \frac{[x_1^i]^2}{2} \tag{113}$$

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Figure 4: Nested Switching Control of a Single Axis of a Magnetic Bearing

$$\dot{V}_1 = x_1^i x_2^i \tag{114}$$

$$= x_1^i[-k_1 sgn[x_1^i]] \text{ in finite time} \tag{115}$$

$$\leq k_1|x_1^i| \tag{116}$$

$$\leq 0 \tag{117}$$

It is clear that $x_1^i \to 0$ in finite time. However, when $x_1^i = 0$, $x_2^i \in [-k_1, k_1]$, and is not equal to 0. This is where chatter commences, and the system limit cycles between the surfaces $x_1^i = 0$ and $x_2^i = k_1 sgn[x_1^i]$. Such limit cycling behaviour is present as the gain k_1 is not slowly reduced as $x_1^i \to 0$. Indeed if the multiplicand of $sgn[x_1^i]$ was to decrease in magnitude and finally equal 0 when $x_1^i = 0$, we can expect x_2^i to also be equal to 0 without chatter. This indeed is the principle behind using saturation functions as opposed to sgn functions in nested control. We will now show an extension of this method, without using saturation functions.

We now try to eliminate the problem of limit cycling between switching surfaces that was mentioned earlier. We do this using the switching control law mentioned earlier which is of the form.

$$u_{switching} = -k_2 sgn[x_2^i + k_1|x_1^i|^{\frac{1}{m}} sgn[x_1^i]] \tag{118}$$

Denote $S = x_2^i + k_1|x_1^i|^{\frac{1}{m}} sgn[x_1^i]$. Note that S is not differentiable at $x_1^i = 0$. However, almost everywhere, the derivative of S may be written as

$$\dot{S} = -k_2 sgn[S] + k_1 \frac{x_2^i}{|x_1^i|^{1-\frac{1}{m}}} \tag{119}$$

By choosing a large value of k_2, we hope to swamp the term $k_1 \frac{x_2^i}{|x_1^i|^{1-\frac{1}{m}}}$. Indeed, only in cases when this is possible, it is possible to conclude that

$$x_2^i = -k_1|x_1^i|^{\frac{1}{m}} sgn[x_1^i] \tag{120}$$

And the conclusions of the previous section follow, without the limit cycle behaviour.

The phase portrait shown below illustrates the properties of the control law. The values of chosen gains are

$$k_1 = 0.5 \tag{121}$$

$$k_2 = 5 \tag{122}$$

$$m = 2 \tag{123}$$

For the same values of control gains, it is possible to choose a higher order fractional index, and the resulting phase portrait is shown below.
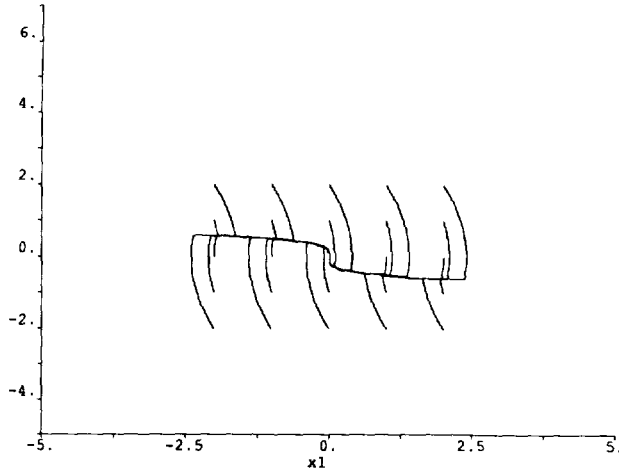
38

Figure 5: Nested Higher Order Switching Control of a Single Axis of Magnetic Bearing

8 Synchronous Sliding Control of the Magnetic Bearing

In this section, we present an interesting property of a modified vector sliding mode control law [AKP92a], [AKP92b], [AKP 5], and its possible application. The property of this modified vector sliding mode control law is such that it achieves simultaneous regulation for a group of n scalar systems with n inputs [SMSar], [SMS93], [SMS 7].This control technique has interesting implications for the magnetic bearing. Using this control technique it is possible to regulate the states of the multivarious magnetic bearings to the origin synchronously thus eliminating the possibility of inducing overshoot torques. The control law has the interesting property that it is a closed loop control law which can be prescribed without *explicit* reference to the initial conditions of the system. The law is interesting in that it introduces coupling between decoupled systems to achieve the synchronization objective. We present the basic theory of synchronous sliding control, and later specialize it to the case of the magnetic bearing.

8.1 Synchronous Sliding

Consider a group of n scalar decoupled systems of the form

$$
\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}
\tag{124}
$$

$$
\begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ \vdots \\ x_{n0} \end{bmatrix}
\tag{125}
$$

where the states $x_i \in \Re$, the controls $u_i \in \Re$ $i = 1, 2, \ldots, n$ the initial conditions $x_{i0} \in \Re$ $i = 1, 2, \ldots, n$. With minor abuse of notation, we create a new state vector $x \in \Re^n$, where $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$

The control objective is to regulate the states from non-zero initial conditions to the origin, in finite time. That is, that there exist instants of time $t_i^* < \infty \in \Re_+$ $i = 1, 2, \ldots, n$ such that the following is true.

$$
x_i(t) = 0 \ \forall \ t \geq t_i^* \ i = 1, 2, \ldots, n
\tag{126}
$$

We choose n sliding mode control laws of the following form to ensure achievement of the control objective.

$$
u_i = -k_i \frac{x_i}{|x_i|} \quad \text{if } |x_i| \neq 0 \ i = 1, 2, \ldots, n
\tag{127}
$$

where $k_i \in \Re_+$

Comment 8.1 We note here that the controls u_i $i = 1, 2, \ldots$ are decoupled, in that u_i is a function only of x_i. Also note that the time taken by each state x_i $i = 1, 2, \ldots, n$ to reach the origin is a function of its initial value $x_i(0)$ $i = 1, 2, \ldots, n$ and the control gains k_i $i = 1, 2, \ldots, n$.

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Comment 8.2 Also note in equation (127) we did not specify the control law at $|x_i| = 0$ $i = 1, 2, \ldots, n$. Indeed, $u_i = -k_i sgn[x_i]$ $|x_i| \neq 0$. We do not specify the control at $|x_i| = 0$. As the control is not specified only on sets of zero measure, it does not affect the existence of Filippov solutions shown by modelling the system by a differential inclusion.

We now present some interesting properties of a modified sliding mode control law that deliberately introduces coupling between the decoupled systems. We present proof of existence of solutions, proof of stability, and proof of synchronous finite time convergence for the modified sliding mode control law. In order to do so, we formalize the notion of *synchronous finite time convergence*.

Definition 8.1 *A set of $n \in Z_+$ variables $x_i(t) : \Re_+ \to \Re$ $i = 1, 2, \ldots, n$ are said to reach the origin synchronously commencing from nonzero initial conditions $x_i(0) \neq 0$ $i = 1, 2, \ldots, n$ if there exists an instant of time $t^* < \infty \in \Re_+$ such that the following is true.*

$$x_i(t) \neq 0 \, \forall \, t < t^* \tag{128}$$

$$x_i(t) = 0 \, \forall \, t \geq t^* \tag{129}$$

$$i = 1, 2, \ldots, n \tag{130}$$

That is to say, that the states with nonzero initial conditions (an assumption we make without loss of generality) are regulated to 0 at the *same* instant of time t^*. There are many practical applications where such *synchronous* regulation is important. A typical application is a multifingered robot hand that grips an object. It is important to ensure that the fingers touch the object *synchronously* and thus cause force closure without imparting motion to the object. We will say more about this later.

It is possible to ensure synchronous motion using a simple sliding mode feedback where the control gains are chosen with explicit dependence on initial conditions. Indeed, given the initial conditions exactly, we choose a decoupled control law that uses the values of initial conditions to derive control gains that guarantee synchronous reaching of the origin. For the sake of completeness we state the control law as follows.

Theorem 8.1 *Synchronous regulation with explicit dependence on initial conditions.*

Given *(G1) A nonlinear system of the form (124) - (125).*

(G2) A control law of the form (127)

If *(I1) k_i $i = 1, 2, \ldots, n$ are chosen such that*

$$\frac{|x_i(0)|}{k_i} = \frac{|x_j(0)|}{k_j} \quad i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, n \tag{131}$$

Then *(T1) Filippov solutions exist for the system (124) - (125) subject to the control law (127).*

(T2) The surfaces $x_i = 0$ $i = 1, 2, \ldots n$ are reached synchronously at a time $t^* = \frac{x_1(0)}{k_1}$.

Proof: ♠ ▷ The proof is quite straightforward and utilizes standard facts from sliding mode control theory.

The existence of Filippov solutions is shown using the fact that the modelling differential inclusions $\mathcal{F}_i(x) : \Re \to [-1, 1]$ are closed, bounded, convex and uppersemicontinuous. Note that $\mathcal{F}_i(x) : \Re \to [-1, 1]$ are defined as follows

$$\mathcal{F}_i(x) = -k_i \frac{x_i}{|x_i|} \text{ if } |x_i| \neq 0 \tag{132}$$

$$\in [-1, 1] \text{ if } |x_i| = 0 \tag{133}$$

Stability is shown using the candidate Lyapunov function $V(x) : \Re^n \to \Re_+$ given by $V(x) = \sum_{i=1}^{n} \frac{x_i^2}{2}$ whose derivative along the flow of (124) - (125) is given by $\dot{V} = -\sum_{i=1}^{n} |x_i|$. Indeed $\dot{V}$ is negative definite proving global exponential stability of the origin.

Finally, the time taken to reach the origin is given by $t_i^* = \frac{|x_i(0)|}{k_i}$ $i = 1, 2, \ldots, n$. Now using the assumption that $\frac{|x_i(0)|}{k_i} = \frac{|x_j(0)|}{k_j}$ $i = 1, 2, \ldots, n$ $j = 1, 2, \ldots, n$, we see that $t_1^* = t_2^* \ldots = t_n^* = t^*$.

This completes the proof of the theorem. ◁ ♠

Comment 8.3 The control law is inelegant to implement as it explicitly depends on the initial conditions. It would be desirable to develop a state feedback control law that would achieve the same objective, but one whose control gains do not explicitly depend on initial conditions.

We now propose a state feedback control law that would ensure synchronous regulation.

Theorem 8.2 *Synchronous regulation with state feedback.*

Given *(G1) A nonlinear system of the form (124) - (125).*

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If *(I1)* The controls u_i $i = 1, 2, \ldots, n$ in equations *(124)* - *(125)* are chosen to be

$$u_i = -k^\star \frac{x_i}{||x||_2} \quad if \ ||x||_2 > 0 \quad i = 1, 2, \ldots, n \tag{134}$$

$$||x||_2 = [\sum_{i=1}^{n} x_i^2]^{\frac{1}{2}} \tag{135}$$

$$\tag{136}$$

where $k^\star \in \Re_+$

Then *(T1)* Filippov solutions exist for the system *(124)* - *(125)* subject to the control law *(134)*.

(T2) The surfaces $x_i = 0$ $i = 1, 2, \ldots n$ are reached synchronously at a time $t^\star = \frac{||x(0)||_2}{k^\star}$ where $||x(0)||_2$ is the 2-norm of the vector of initial conditions, given by $||x(0)||_2 = [\sum_{i=1}^{n} x_i^2(0)]^{\frac{1}{2}}$

Proof: ♠ ▷ We prove the theorem in three steps. First we show existence of generalized Filippov solutions to the system (124) - (125) subject to the control law (134). We then show attractivity of the origin when subject to the control law using a simple Lyapunov argument. Finally we show the achievement of synchronous regulation, by explicitly computing the times taken to reach the origin. We first make the following comments.

Comment 8.4 It is interesting to compare the control laws given by equations (127) and (134). While the control specified by (127) decouples the system entirely, the control specified by (134) introduces a coupling between the through the 2-norm of the state vector $||x||_2$. Furthermore, note that the control gains $k^\star$ remain the same for all u_i $i = 1, 2, \ldots, n$.

Comment 8.5 The discontinuous control law (134) is not defined at the origin, the same way the function $sgn[(\cdot)]$: $\Re \to [-1, 1]$ is not defined when $(\cdot) = 0$. But also note that the control law specified by (134) is bounded by $k^\star$. Indeed, as $\frac{x_i}{||x||_2} \leq 1$ $i = 1, 2, \ldots, n$, $u_i \leq k^\star$ $i = 1, 2, \ldots, n$.

Step 1: Existence Of Filippov Solutions

To show the existence of generalized Filippov solutions we model the system (124) - (125) subject to the control law (134) by the following differential inclusion.

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} \in \begin{bmatrix} \mathcal{F}_1(x) \\ \vdots \\ \mathcal{F}_n(x) \end{bmatrix} \tag{137}$$

where the inclusions $\mathcal{F}_i(x) : \Re \to [-k^\star, k^\star]$ are specified as

$$\mathcal{F}_i(x) = -k^\star \frac{x_i}{||x||_2} \quad \text{if} \ ||x||_2 > 0 \tag{138}$$

$$\in [-k^\star, k^\star] \ \text{if} \ ||x||_2 = 0 \tag{139}$$

$$i = 1, 2, \ldots, n \tag{140}$$

The inclusions $\mathcal{F}_i(x)$ $i = 1, 2, \ldots, n$ are

- closed, bounded, convex and uppersemicontinuous.

Invoking the theorem on the existence of generalized Filippov solutions, we conclude that Filippov solutions exist for the system (124) - (125) subject to the control law (134).

Step 2: Attractivity Of The Origin

Consider a candidate Lyapunov function $V(x) : \Re^n \to \Re_+$ given by

$$V = \frac{x^T x}{2} \tag{141}$$

Differentiating V along the flow of (124) - (125) subject to the control law (134), we find

$$\dot{V} = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} \begin{bmatrix} -k^\star \frac{x_1}{||x||_2} \\ \vdots \\ -k^\star \frac{x_n}{||x||_2} \end{bmatrix} \tag{142}$$

$$= -k^\star \frac{||x||_2^2}{||x||_2} \tag{143}$$

$$= -k^\star ||x||_2 \ \text{if} \ ||x||_2 \neq 0 \tag{144}$$

$$\leq 0 \tag{145}$$

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Negative definiteness of $\dot{V}$ confirms the global exponential stability of the origin.

Step 3: Synchronous Reaching

From system (124) - (125) subject to the control law (134) the following is true for any i, j

$$\dot{x}_i = -k^{\star} \frac{x_i}{||x||_2} \tag{146}$$

$$\dot{x}_j = -k^{\star} \frac{x_j}{||x||_2} \tag{147}$$

$$\frac{dx_i}{dx_j} = \frac{x_i}{x_j} \tag{148}$$

$$\forall \, i, j \leq n \quad i \neq j \, ||x||_2 \neq 0 \tag{149}$$

Solving (148), we obtain explicit expressions for constraints on state trajectories as

$$x_i(t) = \frac{x_i(0)}{x_j(0)} x_j(t) \forall \, i, j \leq n \quad i \neq j \, ||x||_2 \neq 0 \tag{150}$$

Using (150) in (146), we recast (146) in the form

$$\dot{x}_i = -k^{\star} \frac{x_i}{||x||_2} \tag{151}$$

$$= -k^{\star} \frac{x_i}{[\sum_{k=1}^{n} x_k^2]^{\frac{1}{2}}} \tag{152}$$

$$= -k^{\star} \frac{x_i}{[x_i^2 + \sum_{k=1, \, k \neq i}^{n} x_k^2]^{\frac{1}{2}}} \tag{153}$$

$$= -k^{\star} \frac{x_i}{[x_i^2 + \sum_{k=1, \, k \neq i}^{n} \frac{x_k^2(0)}{x_i^2(0)} x_i^2]^{\frac{1}{2}}} \tag{154}$$

$$= -k^{\star} \frac{x_i}{x_i[1 + \sum_{k=1, \, k \neq i}^{n} \frac{x_k^2(0)}{x_i^2(0)}]^{\frac{1}{2}}} \tag{155}$$

$$= -k^{\star} \frac{x_i(0)}{[x_i^2(0) + \sum_{k=1, \, k \neq i}^{n} x_k^2(0)]^{\frac{1}{2}}} \tag{156}$$

$$\dot{x}_i = -k^{\star} \frac{x_i(0)}{||x(0)||_2} \quad i = 1, 2, \ldots, n \tag{157}$$

The righthandside of (157) is a real constant, and therefore the solution of (157) is given by

$$x_i(t) = -k^{\star} \frac{x_i(0)}{||x(0)||_2} t + x_i(0) \quad i = 1, 2, \ldots, n \tag{158}$$

From (158), we obtain the time $t^{\star}$ taken by $x_i(t)$ $i = 1, 2, \ldots, n$ to reach the origin, starting from arbitrary nonzero initial conditions by setting the righthand side of (158) to 0.

$$0 = -k^{\star} \frac{x_i(0)}{||x(0)||_2} t^{\star} + x_i(0) \tag{159}$$

$$t^{\star} = \frac{||x(0)||_2}{k^{\star}} \quad i = 1, 2, \ldots, n \tag{160}$$

Synchronous convergence of state trajectories commencing from nonzero initial conditions is thus shown. This concludes the proof of the theorem. ◁♠

8.2 Design Of Tracking Control Laws

The control laws that we have developed are discontinuous. As a prelude to presenting tracking control laws that involve discontinuities, let us analyze a simple linear pole-placement control law from another perspective. Consider a system represented as a chain of integrators of the form,

$$\dot{x}_1 = x_2 \tag{161}$$

$$\vdots \quad \vdots \quad \vdots \tag{162}$$

$$\dot{x}_{n-1} = x_n \tag{163}$$

$$\dot{x}_n = u \tag{164}$$

where the state vector $x \in \Re^n$ and the control input $u \in \Re$. Given a desired smooth trajectory $x_{1d}(t) : \Re_+ \rightarrow \Re$ to be tracked by the state x_1 we present a tracking control law that uses successive derivative of desired trajectories. We define recursively, a set of desired trajectories for the states as

$$x_{id}(t) = \frac{dx_{i-1,d}(t)}{dt} - k_{i-1}[x_i(t) - x_{i,d}(t)] \quad i = 2, 3, \ldots, n \tag{165}$$

While we are *given* a desired trajectory to be tracked by the state $x_1(t)$, we define desired trajectories for the remaining states the tracking of which automatically ensures the original tracking objective for $x_1(t)$. Indeed, the intuition behind such a definition of desired trajectories becomes clear when we look at $x_{2d}(t)$.

$$x_{2d}(t) = \frac{dx_{1d}(t)}{dt} - k_1[x_1(t) - x_{1d}(t)] \tag{166}$$

From (166) it is clear that when the surface $x_2 = x_{2d}$ the resulting dynamics for $x_1(t)$ is given as

$$\dot{x}_1(t) = x_2(t) \tag{167}$$

$$= x_{2d}(t) \tag{168}$$

$$= \frac{dx_{1d}(t)}{dt} - k_1[x_1(t) - x_{1d}(t)] \tag{169}$$

The dynamics of the system is such as to ensure that $x_1(t) \rightarrow x_{1d}(t)$ exponentially. However, if the surface $x_2 - x_{2d} = 0$ can only be reached exponentially, then the dynamics of $\dot{x}_1$ is perturbed by an exponentially decaying signal, and therefore invoking the result on the exponentially stable systems perturbed by exponentially decaying perturbations, we conclude exponential convergence of $x_1(t)$ to $x_{1,d}(t)$. We now show the relationship between control laws developed using the recursively defined desired trajectories and the standard pole-placement control law.

Theorem 8.3 *Connection between pole-placement and recursive trajectory definition.*

Given *(G1) A nonlinear system of the form (161) - (164).*

(G2) Given a set of desired trajectories of the form (165)

If *(I1) The controls u in equation (164) are chosen to be*

$$u = \frac{dx_{nd}(t)}{dt} - k_n[x_n(t) - x_{nd}(t)] \tag{170}$$

where $x_{i,d}(t) : \Re^n \times \Re_+ \rightarrow \Re \ i = 2, 3, \ldots, n$ is specified by (165) and $k_n \in \Re_+$.

Then *(T1) The control law specified by (170) is a stable pole-placement control with the n eigenvalues each being equal to $-k_i \ i = 1, 2, \ldots, n$*

Proof: ♠ ▷ The proof is obvious by writing the dynamics for $\dot{x}_1$ and $\dot{x}_2$. Indeed,

$$\dot{x}_1 = x_2 \tag{171}$$

$$\dot{x}_2 = \frac{dx_{2d}(t)}{dt} - k_2[x_2(t) - x_{2d}(t)] \tag{172}$$

Using the definition of $x_{2d}(t)$ provided by (165), we rewrite (172) as

$$\dot{x}_1 = x_2 \tag{173}$$

$$\dot{x}_2 = \frac{d[\frac{dx_{1d}(t)}{dt} - k_1[x_1(t) - x_{1d}(t)]]}{dt} \tag{174}$$

$$-k_2[x_2(t) - [\frac{dx_{1d}(t)}{dt} - k_1[x_1(t) - x_{1d}(t)]]] \tag{175}$$

Which may be rewritten as

$$\dot{x}_1 = x_2 \tag{176}$$

$$\dot{x}_2 = \frac{d^2 x_{1d}(t)}{dt^2} + [k_1 + k_2]\frac{dx_{1d}(t)}{dt} + [k_1 k_2]x_{1d}(t) \tag{177}$$

$$-[k_1 + k_2]x_2(t) - [k_1 k_2]x_1(t) \tag{178}$$

That is to say

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{d^2 x_{1d}(t)}{dt^2} + [k_1 + k_2]\frac{dx_{1d}(t)}{dt} + [k_1 k_2]x_{1d}(t) \end{bmatrix} \tag{179}$$

$$- \begin{bmatrix} 0 & 1 \\ -[k_1 k_2] & -[k_1 + k_2] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{180}$$

The placement of poles through recursive trajectory definition is trivially obvious by inspection of equation (180). This concludes the proof of the theorem. ◁ ♠

Comment 8.6 It is to be noted that this tracking control law is valid for any specification of desired trajectories that are smooth, the tracking of which guarantees achievement of the control objective. That is, we are free to specify any smooth set of trajectories $x_{id}(t)$ $i = 2, 3, \ldots, n$, the only constraint being $x_i(t) = x_{id}(t)$ $i = 2, 3, \ldots, n$ $\Rightarrow$ $x_{i-1}(t) \to x_{i-1,d}(t)$. Indeed, the linear pole-placement control law is just a special case of control laws that achieve this tracking objective.

Comment 8.7 We now ask if it is possible to relax the smoothness assumption on the desired trajectories $x_{id}(t)$. Indeed, the first relaxation would be to consider desired trajectories that are differentiable almost everywhere, except possibly on sets of zero measure. The Nested and Switching control laws presented in the previous chapter are examples of such discontinuous control laws, the discontinuities existing on sets of zero measure. The proofs of such control laws are much harder in general, though the regularization of such control laws that involve saturation functions have been used in the recent literature. We have been inspired by the attempts of [Tee92] in developing control laws that use Filippov averaging instead of regularization. That is to say, that we are prepared to tolerate chatter and limit cycling by using discontinuous control laws. The drawback however is that we can show finite time synchronous stabilization only on the average, whereas a regularized control law, by eliminating the discontinuity would permit smooth stabilization, though exponentially, without the chatter.

Comment 8.8 Our interest in relaxing the smoothness assumption on the desired trajectories merely enables us to utilize the discontinuous, synchronous control law for a practical mechanical system.

We will first present the control law for a group of $n \in Z_+$ mechanical systems, and then apply it to a well known example of the magnetic bearing. Many mechanical systems are represented by Newtons force and torque balance equations that assume the form

$$\dot{x}_1^i = x_2^i \tag{181}$$
$$\dot{x}_2^i = v^i \tag{182}$$

where $x^i \in \Re^2$ is the state of the ith mechanical system where $i \leq n \in Z_+$, and $v^i(x, t) : \Re^2 \times \Re_+ \to \Re$ is the input force. Typically, x_1^i represents the generalized position coordinate of the mechanical system, and x_2^i represents the generalized velocity coordinate. These equations, though simple in form, serve to illustrate the application of the theory, and also represent a large class of useful physical systems. Given desired trajectories $x_{1d}^i(t) : \Re_+ \to \Re$ to be tracked by the states $x_1^i(t)$, we attempt to find control laws u^i that ensure synchronous tracking for the states $x_1^i(t)$.

We now state the theorem that ensures synchronous tracking for the systems of the form (181)-(182).

Theorem 8.4 *Synchronous tracking for a class of mechanical systems.*

Given *(G1)* *n mechanical systems, each of the form (181) - (182).*

(G2) *Given a set of desired trajectories of the form $x_{1d}^i(t) : \Re_+ \to \Re$ $i = 1, 2, \ldots, n$*

If *(I1)* *The controls $u^i(x, t)$ $i = 1, 2, \ldots, n$ in equation (182) are chosen to be*

$$v^i = \frac{dx_{2d}^i}{dt} - k_2^\star \frac{x_2^i - x_{2d}^i}{[\sum_{j=1}^n [x_2^j - x_{2d}^j]^2]^{\frac{1}{2}}} \tag{183}$$

$$x_{2d}^i = \frac{dx_{1d}^i}{dt} - k_1^\star \frac{x_1^i - x_{1d}^i}{[\sum_{j=1}^n [x_1^j - x_{1d}^j]^2]^{\frac{1}{2}}} \tag{184}$$

where$k_1, k_2 \in \Re_+$.

Then *(T1)* *Filippov solutions exist for system (181) - (182) subject to control (183).*

(T2) *States $x_1^i(t)$ track their respective trajectories $x_{1d}^i(t)$ synchronously.*

Proof: ♠ ▷ The proof is simple once we realize the validity of the system equations (181) - (182) subject to the control law (183) for arbitrarily small neighborhoods of the origin. Indeed, the control law is undefined only on a set of zero measure. As this set of zero measure is indeed the set we desire to make invariant, and the control law directs system trajectories to this set, and hence maintains invariance, the conclusions of the theorem naturally follow. The theorem can also be proved invoking the results of the nested and switching control laws mentioned in the previous section. ◁ ♠

8.3 Application to the Magnetic Bearing

In this subsection we apply the proposed tracking control law to magnetic bearings.

The dynamics of magnetic bearings are given by the following equations.

$$\dot{x}_1^i = x_2^i \tag{185}$$
$$\dot{x}_2^i = v^i \tag{186}$$
$$i = 1, 2, \ldots, 4 \tag{187}$$

Given desired trajectories $x_{1d}^i(t)$ $i = 1, 2, \ldots, 4$ to be tracked by the respective state variables $x_1^i(t)$ $i = 1, 2, \ldots, 4$, We now define the following set of vectors.

$$e_j^i(t) = x_j^i(t) - x_{jd}^i(t) \quad i = 1, 2, \ldots, 4 \quad j = 1, 2 \tag{188}$$

$$e_1(t) = \begin{bmatrix} e_1^1(t) & \cdots & e_1^4(t) \end{bmatrix}^T \tag{189}$$

$$e_2(t) = \begin{bmatrix} e_2^1(t) & \cdots & e_2^4(t) \end{bmatrix}^T \tag{190}$$

where

$$x_{2d}^i(t) = \frac{dx_{1d}^i(t)}{dt} - k_1^\star \frac{x_1^i(t) - x_{1d}^i(t)}{\|e_1(t)\|} \quad \text{if } \|e_1(t)\|_2 \neq 0 \tag{191}$$

$$= \frac{dx_{1d}^i(t)}{dt} \quad \text{if } \|e_1(t)\|_2 = 0 \tag{192}$$

Now note that $x_{2d}^i(t)$, is not strictly differentiable at the origin, but has a derivative that exists almost everywhere. Indeed define the generalized derivative as

$$\dot{x}_{2d}^i(t) = \frac{dx_{2d}^i(t)}{dt} - \frac{k_1^\star N_1^i(t)}{\|e_1(t)\|^{\frac{3}{2}}} \quad \text{if } \|e_1(t)\|_2 \neq 0 \tag{193}$$

$$= \frac{dx_{2d}^i(t)}{dt} \quad \text{if } \|e_1(t)\|_2 = 0 \tag{194}$$

where

$$N_1^i(t) = \sum e_1^j(t)[e_1^j(t)\dot{e}_1^i(t) - \dot{e}_1^j(t)e_1^i(t)] \quad j = 1, 2, 3, 4 \quad j \neq i \tag{195}$$

$$\dot{e}_1^i(t) = x_2^i(t) - \dot{x}_{id}^i(t) \quad i = 1, 2, \ldots, 4 \tag{196}$$

We now choose v^i $i = 1, 2, \ldots, 4$ in the following manner.

$$v^i(t) = \dot{x}_{2d}^i(t) - k_2^\star \frac{x_2^i(t) - x_{2d}^i(t)}{\|e_2(t)\|} \quad \text{if } \|e_2(t)\|_2 \neq 0 \tag{197}$$

$$= \frac{dx_{1d}^i(t)}{dt} \quad \text{if } \|e_2(t)\|_2 = 0 \tag{198}$$

Claim 8.1 *Synchronous Tracking for a Magnetic Bearing.*

Given *(G1)* *Mechanical systems, each of the form (187).*

(G2) *Given a set of desired trajectories of the form $x_1^i d(t) : \Re_+ \to \Re$ $i = 1, 2, \ldots, n$*

If *(I1)* *The controls $v^i(x, t)$ $i = 1, 2, \ldots, n$ in equation (187) are specified by (197). where $k_1, k_2 \in \Re_+$.*

Then *(T1)* *Filippov solutions exist for system (187) - subject to control (197).*

(T2) *States $x_1^i(t)$ track their respective trajectories $x_{1d}^i(t)$ synchronously.*

Proof: ♠ ▷ The proof of the claim is by invoking the theorem proved earlier for the more general case of a group of mechanical systems.

Indeed, it is easily seen that the application of control (197) would cause the states $x_2^i(t)$ $i = 1, 2, \ldots, 4$ to reach their desired values in finite time, and the desired trajectories are so chosen that the reduced dynamics ensures finite time tracking for $x_{1d}^i(t)$. ◁ ♠

Results of simulation are shown for the following conditions. The chosen desired trajectories were as follows. $x_{1d}^1(t) = \sin t$, $x_{1d}^2(t) = 5$, $x_{1d}^3(t) = -2$, $x_{1d}^4(t) = 5$. The initial conditions were as follows $x_1^1(0) = 1$, $x_1^2(0) = 7$, $x_1^3(0) = -1$, $x_1^4(0) = 2$, $x_2^1(0) = 0$, $x_2^2(0) = 0$, $x_2^3(0) = 0$, $x_2^4(0) = 0$.

Simulation results are in excellent agreement with the predicted behaviour. Indeed, note that the trajectory errors vanish identically at the same instant of time. This indeed was the motivation for considering the synchronous tracking control law.
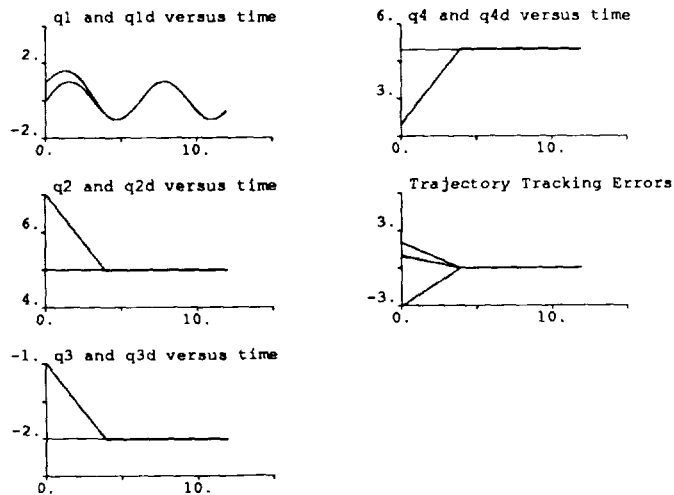
Figure 6: Synchronous Control of a Magnetic Bearing

9 Fractional Control of the Magnetic Bearing

9.1 Introduction

In this section, we will present an interesting variable structure control law for a vector dynamical system, that is a bounded control law, but whose convergence rate is faster than a comparable linear control law, and whose robustness properties are much better than comparable linear control laws [Pra92]. We will clarify what we mean by *comparable linear control laws* in the following subsections. We use the term fractional control law to indicate that this is a particular form of variable structure control law where the powers of indices are positive fractions.

We will present qualitative arguments for the conjecture, and will provide simulation results that are in agreement with the conjecture. However the proof of this conjecture has been quite elusive, and we have been unable to present anything more tangible than this conjecture. We leave the proof of this control method as an open problem to the reader.

9.2 Finite Time With Continuous Control - Scalar Systems

Consider a scalar dynamical system of the form

$$\dot{x} = u \tag{199}$$

where $x \in \Re$ and the control $u \in \Re$. Given the control objective of regulating the state of the system (199) to the origin commencing from arbitrary initial conditions in finite time, we choose u in the following manner.

$$u = -k|x|^{\frac{1}{r}} sgn[x] \tag{200}$$

where $k \in \Re_+$ and $r > 1$.

Comment 9.1 The choice of u is novel since the control is obviously continuous, but not differentiable at the origin. Also note that the control law involves raising the power of $|x|$ to a fraction, and hence the term fractional control.

We now make the following claim regarding existence of trajectories, stability and convergence for the system (199).

Claim 9.1 *Existence of solutions, stability and convergence for fractional control of scalar systems.*

Given

(G1) System dynamics of the form (199)

If

(I1) The control u is specified as in (200)

Then

(T1) Cauchy solutions exist for (199) subject to (200).

(T2) $x = 0$ *is stable.*

(T3) *Indeed* $x \to 0$ *in finite time* $t^{\star}$*, given by* $t^{\star} = \frac{|x(0)|^{1-\frac{1}{r}}}{k[1-\frac{1}{r}]}$

Proof: ♠ ▷ Existence of Cauchy solutions is easily seen by the fact that the righthand side of the differential system is continuous.

Considering the candidate Lyapunov function $V(x) : \Re \to \Re_+$ given by

$$V = \frac{x^2}{2} \tag{201}$$

Indeed $\dot{V} = -k|x|^{1+\frac{1}{r}} \leq 0$. Attractivity of the origin is therefore confirmed.

To show finite time convergence we solve the equation

$$\dot{x} = -k|x|^{\frac{1}{r}} sgn[x] \tag{202}$$

to obtain that $t^{\star} = \frac{|x(0)|^{1-\frac{1}{r}}}{k[1-\frac{1}{r}]}$. The proof of the claim is complete. ◁ ♠

We now make a comparison between three kinds of control laws that regulate the state of the system (199) to the origin.

$$u_{linear} = -kx \tag{203}$$

$$u_{sliding} = -\frac{k}{|x|}x \quad \text{if } |x| > 0 \tag{204}$$

$$u_{fractional} = -\frac{k}{|x|^{1-\frac{1}{r}}}x \quad \text{if } |x| > 0 \tag{205}$$

Comparison of control efforts reveals something interesting. For all $|x| > 1$, the linear controller has the maximum gain, closely followed by the fractional controller, and the sliding mode controller has the smallest gain. However the situation is reversed when $|x| < 1$. .

Similarly, the times taken to reach the origin from initial conditions $x(0) \neq 0$ are

$$t_{linear} = \infty \tag{206}$$

$$t_{sliding} = \frac{|x(0)|}{k} \tag{207}$$

$$t_{fractional} = \frac{|x(0)|^{1-\frac{1}{k}}}{k[1-\frac{1}{k}]} \tag{208}$$

We now formulate an alternative control law that combines the best of both the linear and the fractional control law to give

$$u^{\star} = -kx \quad \text{if } |x| > 1 \tag{209}$$

$$= -\frac{k}{|x|^p}x \quad \text{if } 0 < |x| \leq 1 \tag{210}$$

$$p > 1 \tag{211}$$

Note that we do not bother to define the control law at the origin.

There is yet another viewpoint as to why this control law does better than a linear control law when $|x| < 1$. The linear control law has an eigenvalue $-k$, and but $u^{\star}$ has an eigenvalue $\frac{-k}{|x|^{\frac{1}{p}}}$ (we use the term eigenvalue very loosely here, since strictly speaking even the term eigenvalues does not make sense in a nonlinear context) that is increasing to ∞ as $|x| \to 0$. Though both control laws are bounded, qualitatively, the fractional control law converges much faster to the origin as seen in the following scalar example.

Example 9.1 *Fractional Control - Scalar Case*
Consider the simple scalar example given by the equations

$$\dot{x} = u \tag{212}$$

Choose

$$u_{linear} = -kx \tag{213}$$

$$u^{\star} = -kx \quad \text{if } |x| > 1 \tag{214}$$

$$= -\frac{k}{|x|^p}x \quad \text{if } 0 < |x| \leq 1 \tag{215}$$

$$k = 2 \tag{216}$$

$$p = 2 \tag{217}$$

It is clear from the simulation plots that the modified fractional control law outperforms the linear control law.

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Now consider a linear system in the controllable canonical form, given by the following equations.

$$\dot{x}_1 = x_2 \tag{218}$$

$$\vdots \quad \vdots \quad \vdots \tag{219}$$

$$\dot{x}_n = u \tag{220}$$

where $x \in \Re^n$, $u \in \Re$.

Now choose the control u to be of the following form

$$u = -k_1 x_1 - k_2 x_2 - \cdots - k_n x_n \quad \text{if } \|x\|_2 > 1 \tag{221}$$

$$= -\frac{k_1}{\|x\|_2^{\frac{n}{r}}} x_1 - \frac{k_2}{\|x\|_2^{\frac{n-1}{r}}} x_2 - \cdots - \frac{k_n}{\|x\|_2^{\frac{1}{r}}} x_2 \quad \text{if } 0 < \|x\|_2 < 1 \tag{222}$$

where

$$\|x\|_2 = \sum_{i=1}^{n} x_i^2 \tag{223}$$

$$r > n \tag{224}$$

$$s^n + k_n s^{n-1} + \cdots + k_1 \text{ is a stable Hurwitz polynomial} \tag{225}$$

We now formulate the following conjecture.

Conjecture 9.1 *Existence of solutions, stability and convergence for fractional control of controllable linear systems.*

Given

(G1) System dynamics of the form (218) - (220)

If

(I1) The control u is specified as in (221) - (225)

Then

(T1) Filippov solutions exist for systems (218) - (220) subject to control (221) - (225)

(T2) $x = 0$ is globally stable

(T3) Indeed $x \to 0$ faster than a comparable linear control law of the form $u_{linear} = -k_1 x_1 - k_2 x_2 - \cdots - k_n x_n$

Qualitative Proof:
First we note that within the unit ball ($\|x\|_2 < 1$), the control effort is bounded by

$$|u| \leq \sum_{i=1}^{n} k_i \tag{226}$$

So the control does not blow up at any instant of time. We have used the notion that in the nonlinear setting, within the unit ball, we have each eigenvalue λ_i $i = 1, 2, \ldots, n$ of this system being replaced by $\frac{\lambda_i}{\|x\|_2^{\frac{1}{r}}}$ where $r > n$.

Consequently, from the way the λ_i $i = 1, 2, \ldots, n$ combine to form the k_i of the control law, the form of the control law is intuitively obvious.

We find by simulation that the robustness, and rate of convergence of the proposed nonlinear law are much superior to a linear control law. The proof of this conjecture, however, has eluded us.

Example 9.2 *Fractional Control for Magnetic Bearing*
We present simulation results for a system of the form

$$\dot{x}_1^i = x_2^i \tag{227}$$

$$\dot{x}_2^i = v^i \tag{228}$$

where

$$v^i = -k_1 x_1^i - k_2 x_2^i \quad \text{if } \|x\|_2 > 1 \tag{229}$$

$$= -\frac{k_1}{\|x\|_2^{\frac{1}{9}}} x_1^i - \frac{k_2}{\|x\|_2^{\frac{2}{3}}} x_2^i \quad \text{if } 0 < \|x\|_2 < 1 \tag{230}$$

$$k_1 = 6 \tag{231}$$

$$k_2 = 11 \tag{232}$$

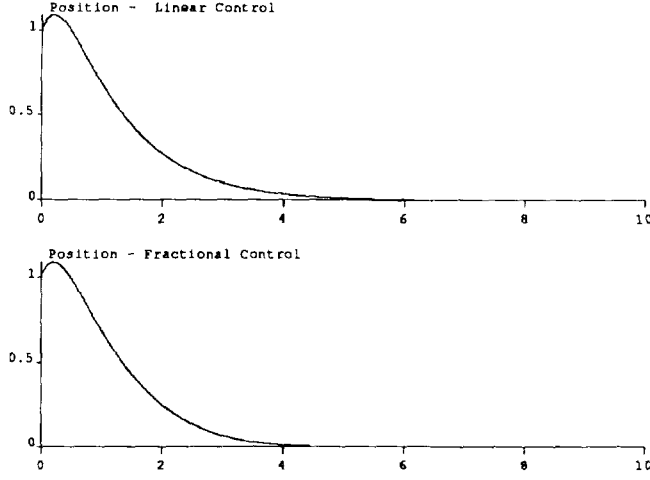*The results show the faster convergence of the state subject to fractional control.*

Figure 7: Fractional Control of a Single Axis of a Magnetic Bearing

10 Sliding Mode Observers for the Magnetic Bearing

10.1 Introduction

We first present the basic theory of sliding mode observers for mechanical systems, and prove the existence of generalized Filippov solutions and stability. We then show the convergence of the observer state errors to zero. We then present the problem with existing theory, and present bounds on variables that would prevent observer failure. Finally we remark on the utilization of the computed bounds as a design rule to help design such sliding mode observers.

The problem of designing observers using sliding mode theory was first introduced and studied by [Mis88]. Here the observation problem is treated as a special case of a state regulation problem. Sliding surfaces are designed based on the error dynamics, and reaching a sliding surface is equivalent to the error in the estimate of the measured state decaying to zero. In sliding mode control, the surface $S = 0$ is reached in finite time, and on that surface the states decay exponentially. Similarly, in sliding mode observer theory, the error in the estimate of the measured state decays in finite time. All other state errors decay exponentially.

Consider a simple mechanical system of the form

$$\dot{x}_1 = x_2 \tag{233}$$
$$\dot{x}_2 = u \tag{234}$$

where $x \in \Re^2$ and $u \in \Re$. Now consider an observer of the following form.

$$\dot{\hat{x}}_1 = \hat{x}_2 + k_1 sgn[\tilde{x}_1] \tag{235}$$
$$\dot{\hat{x}}_2 = k_2 sgn[\tilde{x}_2] \tag{236}$$
$$\tilde{x} = x - \hat{x} \tag{237}$$

Such an observer structure equation leads to error dynamics of the form

$$\dot{\tilde{x}}_1 = \tilde{x}_2 - k_1 sgn(\tilde{x}_1) \tag{238}$$
$$\dot{\tilde{x}}_2 = -k_2 sgn[\tilde{x}_1] \tag{239}$$

Theorem 10.1 *Convergence of the state estimation errors:*

Given

(G1) Error dynamics of the form (238)- (239)

If

(I1) $|\tilde{x}_2| < k_1$

Then

(T1) Generalized Filippov Solutions exist for the system (238)- (239)

(T2) The one-dimensional manifold $\tilde{x}_1 = 0$ is attractive

(T3) The averaged dynamics of $\tilde{x}_2$ about the surface $\tilde{x}_1 = 0$ decays exponentially.

Proof: ♠ ▷ Existence of Filippov solutions is due to the fact that the governing differential inclusions are closed, bounded, convex and uppersemicontinuous.

We will prove the theorem using simple Lyapunov analysis. Consider the candidate Lyapunov function,

$$V = \frac{\tilde{x}_1^2}{2} \tag{240}$$

Differentiating V along the flow of the system(238), we get,

$$\dot{V} = \tilde{x}_1[\tilde{x}_2 + k_1 sgn[\tilde{x}_1]] \tag{241}$$

$$< -||\tilde{x}_1||[k_1 - \tilde{x}_2 sgn[\tilde{x}_1]] \tag{242}$$

Thus as long $\tilde{x}_2 < k_1$, $\dot{V} < 0$, indeed the surface $\tilde{x} = 0$ is attractive.

Comment 10.1 The Theorem asserts the existence of a tubular neighbourhood around the $\tilde{x}_2 = 0$ axis where the trajectories converge to the manifold given by $\tilde{x}_1 = 0$. It is to be noted that $\tilde{x}_2$ must not be greater than k_1 until the trajectories converge to $\tilde{x}_1 = 0$. Some additional conditions are necessary to prevent such an occurrence.

The dynamics of the system when constrained to evolve on the surface $\tilde{x}_1 = 0$, can be derived using the Fillipov solution concept. Thus, taking a convex combination of the dynamics on either side of the sliding surface, we get,

$$\dot{\tilde{x}}_1 = \gamma[\tilde{x}_2 + k_1] + (1 - \gamma)[\tilde{x}_2 - k_1] \tag{243}$$

$$\dot{\tilde{x}}_2 = \gamma k_2 + (1 - \gamma)(-k_2] \tag{244}$$

From the above equations, we eliminate γ, and from the invariance of the sliding surface, we get,

$$\dot{\tilde{x}}_1 = 0 \tag{245}$$

$$\dot{\tilde{x}}_2 = -\frac{k_2}{k_1}\tilde{x}_2 \tag{246}$$

Exponential decay of $\tilde{x}_2$ is clear from the above equation. The proof of the theorem is complete. ◁ ♠

We will now utilize this design technique to design sliding mode observers for the magnetic bearing. Consider the magnetic bearing system represented by the following equations.

$$\dot{x}_1^1 = x_2^1 \tag{247}$$

$$\dot{x}_2^1 = u^1 \tag{248}$$

$$\dot{x}_1^2 = x_2^2 \tag{249}$$

$$\dot{x}_2^2 = u^2 \tag{250}$$

$$\dot{x}_1^3 = x_2^3 \tag{251}$$

$$\dot{x}_2^3 = -a * \omega * x_2^4 + u^3 \tag{252}$$

$$\dot{x}_1^4 = x_2^4 \tag{253}$$

$$\dot{x}_2^4 = a * \omega * x_2^4 + u^1 \tag{254}$$

We design a sliding mode observer for this system of the following form,

$$\dot{\hat{x}}_1^1 = \hat{x}_2^1 + k_1 sgn[\tilde{x}_1^1] \tag{255}$$

$$\dot{\hat{x}}_2^1 = u^1 + k_2 sgn[\tilde{x}_1^1] \tag{256}$$

$$\dot{\hat{x}}_1^2 = \hat{x}_2^2 + k_1 sgn[\tilde{x}_1^2] \tag{257}$$

$$\dot{\hat{x}}_2^2 = u^2 + k_2 sgn[\tilde{x}_1^2] \tag{258}$$

$$\dot{\hat{x}}_1^3 = \hat{x}_2^3 + k_1 sgn[\tilde{x}_1^3] \tag{259}$$

$$\dot{\hat{x}}_2^3 = -a * \omega * \hat{x}_2^4 + u^3 + k_2 sgn[\tilde{x}_1^3] \tag{260}$$

$$\dot{\hat{x}}_1^4 = \hat{x}_2^4 + k_1 sgn[\tilde{x}_1^4] \tag{261}$$

$$\dot{\hat{x}}_2^4 = a * \omega * \hat{x}_2^3 + u^4 + k_2 sgn[\tilde{x}_1^4] \tag{262}$$

where $\tilde{x}_j^i = x_j^i - \hat{x}_j^i$ $i = 1, 2, \dots, 4$ $j = 1, 2$, and $k_1, k_2 > 0$.

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We write the observer error equations as follows.

$$\dot{\tilde{x}}_1^1 = \tilde{x}_2^1 - k_1 sgn[\tilde{x}_1^1] \tag{263}$$

$$\dot{\tilde{x}}_2^1 = -k_2 sgn[\tilde{x}_1^1] \tag{264}$$

$$\dot{\tilde{x}}_1^2 = \tilde{x}_2^2 - k_1 sgn[\tilde{x}_1^2] \tag{265}$$

$$\dot{\tilde{x}}_2^2 = -k_2 sgn[\tilde{x}_1^2] \tag{266}$$

$$\dot{\tilde{x}}_1^3 = \tilde{x}_2^3 - k_1 sgn[\tilde{x}_1^3] \tag{267}$$

$$\dot{\tilde{x}}_2^3 = -a*\omega*\tilde{x}_2^4 - k_2 sgn[\tilde{x}_1^3] \tag{268}$$

$$\dot{\tilde{x}}_1^4 = \tilde{x}_2^4 - k_1 sgn[\tilde{x}_1^4] \tag{269}$$

$$\dot{\tilde{x}}_2^4 = a*\omega*\tilde{x}_2^3 - k_2 sgn[\tilde{x}_1^4] \tag{270}$$

We now state the result concerning the stability and convergence of the observer states to their true values.

Theorem 10.2 *Convergence of the state estimation errors:*

Given

(G1) Error dynamics of the form (263)- (270)

If

(I1) $|\tilde{x}_2^i| < k_1$

Then

(T1) Generalized Filippov Solutions exist for the system (238)- (239).

(T2) The one-dimensional manifold $\tilde{x}_1^i = 0$ is attractive.

(T3) The averaged dynamics of $\tilde{x}_2^i$ about the surface $\tilde{x}_1^i = 0$ decays exponentially.

Proof: ♠ ▷ Stability of the error dynamics is easily shown utilizing the following Lyapunov function.

$$V = \sum_{i=1}^{4} k_2 |\tilde{x}_1^i| + \frac{[\tilde{x}_2^i]^2}{2} \tag{271}$$

$$\dot{V} = -k_1 k_2 \sum_{i=1}^{4} [sgn[\tilde{x}_1^i]]^2 \tag{272}$$

$$\leq 0 \tag{273}$$

Furthermore, $\dot{V} = 0 \rightarrow sgn[\tilde{x}_1^i] = 0$. Invoking the invariance principle of LaSalle, it is seen that the largest invariant set containing the set $\tilde{x}_1^i = 0$ $i = 1.2....,4$ is the set $\tilde{x}_2^i = 0, i = 1,2,...,4$. Stability, and hence convergence to the origin is therefore assured.

We show convergence of the states $\tilde{x}_1^i, i = 1,2,...,4$ to the origin in finite time as follows. As the system is asymptotically stable, there exists an instant of time t^* such that for all $t > t^{star}$, $||\tilde{x}||_2 < k_1$, $\rightarrow |\tilde{x}_2^i| < k_1$. Invoking the theorem on the finite time convergence of the sliding mode observer, the observer states converge in finite time. ◁ ♠

11   Closure

We have shown a variety of nonlinear controllers for the magnetic bearing that are simple and robust to build and are guaranteed to be stable. We believe that the design of controllers utilizing principles of nonlinear analysis provides new richness, insight and excitement in the design of high precision magnetic bearings.

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12 Appendix - Mathematical Foundations of Discontinuous Control

In this section we present basic definitions, facts, and examples about measures of sets and functions, integrability, absolute continuity, convexity, and differential inclusions. For further details about the definitions to follow, refer to [KF70], [AB90], [YCBDB82], [Rud64], [Bar76], [AC84]. We will use the following concepts to develop solutions of differential equations with discontinuous right hand sides.

12.1 Measure of Sets

We commence by formalizing the notion of an interval.

Definition 12.1 *Let $\Re^p$ denote p dimensional Euclidean space. By an **interval** in $\Re^p$, we refer to the set of points $x = [x_1^i, \ldots, x_p]^T$ such that*

$$\exists a_i \leq x_i \leq b_i \quad (i = 1, \ldots, p) \tag{274}$$

The possibility that $a_i = b_i$ is not ruled out, and the empty set is also included as a possible candidate for the intervals. An interval can be understood to refer to a p dimensional cube in $\Re^p$.

Definition 12.2 *If A is the union of a finite number of intervals, then A is said to be an* **elementary set.**

Definition 12.3 *If I is an interval in $\Re^p$, the* **measure** μ of the interval I is defined to be

$$\mu : \Re^p \to \Re = \times_{i=1}^{p}(b_i - a_i) \tag{275}$$

The measure of a set is in some rough sense the volume of the geometric object formed by that set. Indeed, the measure of a $k \leq n-1$ dimensional object in n dimensional space is 0. Therefore, the measure of a point in $\Re^2$ is 0, and the measure of a plane in $\Re^3$ is also 0. The kinds of control we work with will vanish on an $n-1$ dimensional subspace of n dimensional space. Therefore they vanish on a subspace of 0 measure.

Definition 12.4 *If the set A is the union of a finite number of pairwise disjoint intervals, (i.e) $A = \bigcup_{j=1}^{n} I_j$ where $I_j \cap I_k = \emptyset \ \forall \ j \neq k$, then the* **measure** μ of the set A is

$$\mu(A) = \sum_{j=1}^{n} \mu(I_j) \tag{276}$$

Fact 12.4.1 Open sets are measurable.

Fact 12.4.2 The union of a sequence of measurable sets is also measurable.

Fact 12.4.3 The complement of a measurable set is also measurable.

Fact 12.4.4 A set consisting of one point is measurable. Its measure is 0.

Fact 12.4.5 A denumerable set (a union of a sequence of countably many one-point sets) is measurable. Its measure is 0.

Fact 12.4.6 Every subset of a set of measure 0 is measurable. Its measure is 0.

Example 12.1 *Measure of Set of Undefined Control*
  Consider the control $u(t) : \Re_+ \to \Re$ given by

$$u(t) = -sgn[x] \tag{277}$$

where $x \in \Re$. Note that the function $sgn[x] : \Re - \{0\} \to [-1, 1]$ is not defined at 0. Using fact (12.4.4), we assert that the control is not defined on a set of zero measure.

Definition 12.5 *By* **almost everywhere,** *we mean everywhere excepting possibly on a set of measure 0.*

Simply, by saying a relation holds true almost everywhere, we assert that the set of points where the relation fails to hold, has measure zero.

Example 12.2 *Behaviour of Functions*
  Consider functions $f : X \to \Re$, $g : X \to \Re$, $f_n : X \to \Re \ n = 1, 2, \ldots$. The following are the instances of almost everywhere relations between the functions.

  1. $f = g$ almost everywhere, if $\mu\{x \in X : f(x) \neq g(x)\} = 0$.

  2. $f \geq g$ almost everywhere, if $\mu\{x \in X : f(x) < g(x)\} = 0$.

  3. $f_n \to f$ almost everywhere, if $\mu\{x \in X : f_n(x) \neq f(x)\} = 0$.

  4. $f_n \uparrow f$ almost everywhere, if $f_n \leq f_{n+1}$ almost everywhere for all n and $f_n \to f$ almost everywhere.

  5. $f_n \downarrow f$ almost everywhere, if $f_{n+1} \leq f_n$ almost everywhere for all n and $f_n \to f$ almost everywhere.

Example 12.3 *Controls Defined Almost Everywhere*
  Consider the control $u(t) : \Re_+ \to \Re$ given by

$$u(t) = -sgn[x] \tag{278}$$

where $x \in \Re$ Note that the control $u(t)$ is defined almost everywhere.

12.2 Measurable Functions

In the definitions to follow, we will consider real valued functions that map a measurable space X with measure μ to the extended real line $\Re$.

Definition 12.6 *A function $f : X \to \Re$ is measurable if the set $S = \{x \in X : |f(x)| < a\}$ is measurable for all $a > 0 \in \Re$.*

Measurability of a function is a property of the function, based on the measurability of a certain set in its domain.

Example 12.4 *Measurable Functions*

All continuous functions $f : \Re^n \to \Re$ are measurable. Proof is by showing that by definition, S is open, and hence measurable.

The following are some facts based on operations between measurable functions.

Fact 12.6.7 *Given $f : X \to \Re$, if f is measurable, then $\forall k \in \Re$, kf is measurable.*

Fact 12.6.8 *Given $f : X \to \Re$, and $g : X \to \Re$, if f and g are measurable, then $f + g$, $f - g$, and fg are measurable.*

Fact 12.6.9 *Given $p : X \to \Re$, and $f : X \to \Re$, if f is measurable, and if $f(x) = p(x)$ almost everywhere, then p is also measurable.*

We now present a fact concerning the properties of the limit function, based on the properties of the convergent functions.

Fact 12.6.10 *Given a sequence of functions $f_i : X \to \Re$ $i = 1, 2, \ldots$ convergent almost everywhere to the function $f : X \to \Re$, if each $f_i, i = 1, 2, \ldots$ is measurable then the function f is also measurable.*

12.3 Integrable Functions

The key idea of the integral developed by Lesbegue is as follows. In Riemann integration, if $f : [a, b] \in \Re \to \Re$, then to form the Riemann integral we divide the domain of f, $[a, b]$ into many subintervals and group together neighboring points in the domain of the function f. On the other hand, the Lesbegue integral is formed by grouping together points of the domain where the function f takes neighboring values in the *range !* Indeed, the key idea is to partition the range of a function rather than the domain. It is immediately obvious that such a technique allows us to consider functions that have multiple points of discontinuity, or may not even be defined at some points.

The functions in this section map a measurable space X with measure μ into the extended real line $\Re$.

Definition 12.7 *Let $f : X \to \Re$ be a measurable function. Furthermore, let f take no more than countably many distinct values $y_1, y_2, \ldots, y_n, \ldots$ in its range. Then, the Lesbegue integral of the function f over the set A, denoted by $\int_A f(x) d\mu$. is given by*

$$\int_A f(x) d\mu = \sum_j y_j \mu(A_j) \tag{279}$$

where

$$A_j = \{x : x \in A, f(x) = y_j\} \tag{280}$$

if the series (280) is convergent. If the Lesbegue integral of the function f exists, then we say the function is integrable, or summable with respect to the measure μ on the set A.

Example 12.5 *Integrable Functions*

Consider the constant function $f(x) = 1$. Let us evaluate the Lesbegue integral of the function. Indeed,

$$\int_A f(x) d\mu = \int_A 1 d\mu \tag{281}$$

$$= \mu(A) \tag{282}$$

As the Lesbegue integral exists, the function is said to be integrable.

Fact 12.7.11 *Given a measurable function $f : X \to \Re$, if there exists a sequence $f_n : X \to \Re$ of integrable functions converging uniformly to f on the set A, then the function f is said to be integrable.*

Fact 12.7.12 Given $f : X \to \Re$, $g : X \to \Re$, if $g > 0$, and g is integrable on a set A, and $|f(x)| \leq g(x)$ almost everywhere on A, then f is also integrable on A, and

$$\left| \int_A f(x)d\mu \right| \leq \int_A \phi(x)d\mu \tag{283}$$

Fact 12.7.13 Given $f : X \to \Re$, if f is bounded and measurable on a set A, then f is integrable on A.

Fact 12.7.14 Given $f : X \to \Re$, if f is integrable on a set A, then f is integrable on every measurable subset of A.

Fact 12.7.15 Given $f_n : X \to \Re$ $i = 1, 2, \ldots$, a sequence of functions converging to a limit $f : X \to \Re$ almost everywhere on a set A, if there exists a function $g : X \to \Re$ integrable on the set A, and $|f(x)| \leq g(x)$, $\forall x$ almost everywhere in A, then f is integrable on A, and

$$\lim_{n \to \infty} \int_A f_n(x)d\mu = \int_A f(x)d\mu \tag{284}$$

Fact 12.7.16 Given $f_n : X \to \Re$, a sequence of functions converging to a limit $f : X \to \Re$ almost everywhere on a set A, if $\exists k \in \Re$ such that $|f(x)| \leq k$, $\forall x$ almost everywhere in A $n = 1, 2, \ldots$, then f is integrable on A, and

$$\lim_{n \to \infty} \int_A f_n(x)d\mu = \int_A f(x)d\mu \tag{285}$$

12.4 Absolute Continuity

Definition 12.8 *The function $f : X \to \Re$ is said to be absolutely continuous on the interval $[a, b] \in \Re$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for finitely many disjoint open intervals $(a_i, b_i) \subset [a, b]$*

$$\sum_{i=1}^{n} |b_i - a_i| < \delta \Rightarrow \sum_{i}^{n} |f(b_i) - f(a_i)| < \epsilon \tag{286}$$

That is, the function is of bounded variation. It is to be noted, however, that functions of bounded variation need not necessarily be absolutely continuous.

Fact 12.8.17 An absolutely continuous function $f : X \to \Re$ is necessarily continuous.

Note however, that the converse is not true. We will now present an example to illustrate that the converse is not true.

Example 12.6 *Continuity and Absolute Continuity*
Consider the function $f : [0, 1] \to \Re$ defined as follows

$$f(0) = 0 \tag{287}$$
$$f(x) = x^2 \cos \frac{1}{x^2} \quad 0 < x \leq 1 \tag{288}$$

The function f is differentiable at each $x \in [0, 1]$, but is not of bounded variation. This is trivially shown by considering a partitioning as follows.

$$P_n = \{0, \sqrt{\frac{2}{2n\pi}}, \sqrt{\frac{2}{(2n-1)\pi}}, \ldots, \sqrt{\frac{2}{3\pi}}, \sqrt{\frac{2}{2\pi}}, 1\} \tag{289}$$

$$Variation\ of\ f = \cos 1 + \frac{2}{\pi} \sum_{j=1}^{n} \frac{1}{j} \tag{290}$$

$$= \infty \tag{291}$$

The function is not of bounded variation, and hence is not absolutely continuous. We have thus shown an example of a continuous function that is not absolutely continuous.

Fact 12.8.18 If $f : X \to \Re$, then if f is differentiable, the f is absolutely continuous.

Fact 12.8.19 Any function that satisfies the Lipschitz condition is absolutely continuous.

The importance of absolute continuity is that for Lesbegue integration, the fundamental theorem of calculus holds precisely only for absolutely continuous functions.

12.5 Convexity

Definition 12.9 *A set $M \in \Re^n$ is said to be* **convex** *if it contains the joining of any two points in the set.*

Example 12.7 *Set of Bounded Functions*

Consider $C_{[a,b]}$, the space of all continuous real valued functions $f : [a,b] \in \Re \to \Re$, and let M be the subset of $C_{[a,b]}$ defined as follows

$$M = \{f \in C_{[a,b]} : |f(x)| \le 1\} \tag{292}$$

Then, the set M is convex. This is easily evidenced by considering the join of two functions $f(x), g(x) \in M$. For any $\alpha, \beta > 0$ such that $\alpha + \beta = 1$, we have

$$
\begin{aligned}
|\alpha f(x) + \beta g(x)| &\le |\alpha + \beta| \tag{293}\\
&\le 1 \tag{294}\\
&\in M \tag{295}
\end{aligned}
$$

The join of two elements of M belongs to M, and convexity is shown.

Fact 12.9.20 *If a set M is convex, so is its interior.*

Fact 12.9.21 *The intersection of a finite number of convex sets is also convex. That is,*

$$\bigcap_{i=1}^{n} M_i \text{ is convex} \tag{296}$$

where each M_i is convex.

Definition 12.10 *The minimal convex set containing a convex set M is called the* **convex hull** *of M.*

12.6 Set Valued Maps

Definition 12.11 *A function $f(x) : \Re^n \to \Re$ is said to be* **upper semi-continuous** *at a point $x^* \in \Re^n$ if*

$$f(x^*) = \limsup_{x \to x^*} f(x) \tag{297}$$

Example 12.8 *Upper-Semicontinuous sgn Function*

Consider the function $sgn[x] : \Re \to [-1,1]$ defined as follows

$$
\begin{aligned}
sgn[x] &= 1 \quad x \ge 0 \tag{298}\\
&= -1 \quad x < 0 \tag{299}
\end{aligned}
$$

It is obvious that at $x = 0$, the function is upper-semicontinuous.

Definition 12.12 *Having defined some properties of real-valued functions, we now move on to discussing some important properties of set-valued functions. Given two sets X and Y, we define a map $F : X \to Y$ to be* **set-valued**, *if $F : X \to Y$ associates to any $x \in X$, a subset $F(x) \in Y$.*

Definition 12.13 *The* **domain** *of a set valued map $F : X \to Y$ is defined to be $Domain(F) = \{x \in X : F(x) \ne 0\}$ and the* **range** *$Range(F)$ of a set valued map $F : X \to Y$ is defined to be $\bigcup_{x \in X} F(x)$.*

Definition 12.14 *The domain $Domain(F)$ of a set valued map $F : X \to Y$ is defined to be* **strict** *if $Domain(F) = X$, and is defined to be* **proper** *if $Domain(F) \ne 0$.*

Definition 12.15 *A set-valued map $F : X \to Y$ is defined to be* **compact** *if its range $Range(F)$ is a compact subset of Y.*

Definition 12.16 *A set valued map $F : X \to Y$ is* **upper semicontinuous** *at $x^* \in X$, if for any open set S_Y containing $F(x^*)$ there exists a neighbourhood S_X of x^* such that $F(S_X) \subset S_Y$.*

In this section we present basic results for the local existence of solutions of differential equations with discontinuous righthand sides. We define a sliding mode, and present conditions for the existence of a sliding mode. We then present briefly the development of the sliding mode control law, and the various regularizations of it.

We will now state without proof the following two important results from analysis that we will need.

Arzela-Ascoli Theorem:

Let K be a compact subset of R^p and let F be a collection of functions which are continuous on K and have values in R^q. The following properties are equivalent.

1. The family F is uniformly bounded and equicontinuous on K.

2. Every sequence from F has a subsequence which is uniformly convergent on K.

The theorem allows us to define a sequence of approximate solutions of a differential equation, and guarantees convergence of the approximate solutions to a limit function of the sequence is equicontinuous and uniformly bounded.

Filippov Convergence Lemma:

Given a differential inclusion of the form $\dot{x} = \mathcal{F}(x, t)$. If the inclusion $\mathcal{F}(x, t)$ is closed, bounded, convex, and uppersemicontinuous, the limit of any uniformly convergent sequence of approximate solutions of the differential inclusion is also a solution of this inclusion, in the domain of convergence.

That the limit function satisfies the differential inclusion is the main reason for invoking the lemma.