EXTENDED H_2 SYNTHESIS FOR MULTIPLE DEGREE–OF–FREEDOM CONTROLLERS

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SUMMARY

H_2 synthesis techniques are developed for a general multiple–input–multiple–output (MIMO) system subject to both stochastic and deterministic disturbances. The H_2 synthesis is extended by incorporation of anticipated disturbance power–spectral–density information into the controller–design process, as well as by frequency weightings of generalized coordinates and control inputs. The methodology is applied to a simple single–input–multiple–output (SIMO) problem, analogous to the type of vibration isolation problem anticipated in microgravity research experiments.

INTRODUCTION

The vibration environment onboard current and planned manned orbiters requires isolation for microgravity science experiments. The disturbance frequencies are sufficiently low, and the attenuation requirements sufficiently great, so as to preclude a solely passive isolation system (ref. 1).

Since the disturbances to be attenuated are three–dimensional (ref 2, p.2), the isolation actuator must be capable of acting over six degrees of freedom. The requisite multiple–degree–of–freedom (MDOF) controller is much more difficult to design than a single–degree–of–freedom (SDOF) controller, because the isolation system has many inputs (actuator forces) and outputs (measured displacements and accelerations). Multiple–input–multiple–output (MIMO) designs can be very susceptible to unmodeled cross–coupling between channels of input or output (ref. 3), a problem not encountered in SDOF design. The control forces used must therefore be properly coordinated if the controller's performance is to be sufficiently insensitive to unmodeled dynamics (i.e., *robust*). The design of a robust MIMO control system requires the iterative use of synthesis and analysis tools, the former for controller design and the latter for system performance and stability evaluation (ref. 4).

A particular vibration isolation problem may involve different kinds of undesirable outputs, such as excessive absolute accelerations and unacceptable relative displacements. Some of these undesired outputs may be more important than others, and the degree of undesirability may vary with direction or frequency. For example, rattlespace constraints may be highly directional. Or a crystal–growth experiment may be particularly sensitive to accelerations at certain frequencies (ref. 2, p. 7) or in certain directions. One of the goals, then, must be to design a controller capable of minimizing selected plant outputs as dictated by these considerations.

Plant outputs, however, cannot be minimized apart from consideration of the associated control costs, because any active control both consumes power and releases heat. Since both of these costs are of concern in a space environment, the control effort used should not be excessive. And at higher frequencies control effort should also be minimized in order to limit controller bandwidth for the sake of robustness concerns (ref. 5, p. 218).

This paper describes a design procedure, known as extended H_2 synthesis (ref. 5, p. 267), for developing active isolation system controllers. A single–input–multiple–output design problem is then addressed using the presented procedure.

BASIC PROBLEM AND SOLUTION

Problem Statement

We will use Linear Quadratic Gaussian (LQG) theory to design the MDOF controller. This theory has been extensively studied and ussed. LQG is chosen as a synthesis procedure since the quadratic performance index relates well to root–mean–square statistics and power spectral density.

When linearized, the differential equations of motion of the plant can be representable in state–space form by the first order system of equations

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} + E_d\underline{f}_d + E_s\underline{w}_s \tag{1a}$$
$$\underline{y} = C\underline{x} + D\underline{u} \tag{1b}$$
$$\underline{z} = \underline{y} + M\underline{n} \tag{1c}$$

where $\underline{x}$ is the state vector, $\underline{y}$ is the output vector, $\underline{z}$ is the measurement vector, $\underline{u}$ is the control vector, $\underline{f}_d$ is a known or measurable disturbance vector, and $\underline{w}_s$ and $\underline{n}$ are process– and sensor noise respectively. We begin by making a series of reasonable mathematical assumptions. Assume that not all states are accessible, so that rank C $\leq$ dim $\underline{x}$. Let the initial conditions on the state vector be $\underline{x}(0) = \underline{x}_0$; let $\underline{x}_0$, $\underline{w}_s$, $\underline{n}$, and $\underline{f}_d$ be independent and bounded; let $\underline{x}_0$ be Gaussian (ref. 6, p. 272); and let $\underline{n}$ and $\underline{w}_s$ be zero–mean white Gaussian, with $\mathrm{cov}[\underline{w}_s(t),\underline{w}_s(\tau)] = V_1\delta(t-\tau)$ and $\mathrm{cov}[\underline{n}(t), \underline{n}(\tau)] = V_3\delta(t-\tau)$ (ref. 6, p. 272). Assume that $\{A,B\}$ and $\{A,E_s V_1{}^{1/2}\}$ are stabilizable, where $V_1 = V_1{}^{1/2} V_1{}^{1/2*}$ (the asterisk here means "conjugate transpose"); and that $\{C,A\}$ is detectable (ref. 5, p. 226). Let V_1 and V_3 be positive semidefinite (PSD) and positive definite (PD), respectively.

We choose a performance index of the form

$$J = \mathscr{E}\left\{\lim_{T\to\infty} \frac{1}{T} \int_0^T \left\langle [\underline{x}' \ \underline{u}'] \begin{bmatrix} W_1 & W_2 \\ W_2' & W_3 \end{bmatrix} \left\{\frac{\underline{x}}{\underline{u}}\right\} \right\rangle dt \right\} \tag{2}$$

where W_1 is PSD and W_3 is PD (ref. 6; pp. 272, 276). "$\mathscr{E}$" is the expected–value operator, needed since the system is excited stochastically by $\underline{w}_s$. The cost rate functional form (with "$\lim_{T\to\infty} \frac{1}{T}$") is used to allow both for the white noise disturbance $\underline{w}_s$ and for the non–dwindling disturbance $\underline{f}_d$.

If $\underline{Z}(t)$ is defined by $\underline{Z}(t) = \{\underline{z}(\tau), 0 \le \tau \le t\}$; and if $\underline{u}(t) = \underline{\alpha}[t, \underline{Z}(t), \underline{f}_d]$ defines the set of admissible controls (ref. 6, p. 272), where $\underline{\alpha}$ is a vector operator that is linear in terms of its arguments; the basic problem objective is to find an admissible control function $\underline{u}^*(t)$ which minimizes J with respect to the set of admissible control functions $\underline{u}(t)$. [The asterisk here indicates optimality, in the sense defined by Eqn. (2).]

Problem Decomposition

The basic problem, as stated in Eqns. (1) and (2), can be decomposed into two parallel subproblems, one stochastic and the other deterministic. Suppose that $\underline{x} = \underline{x}_s + \underline{x}_d$, where $\underline{x}_s$ is the portion of the system response due to disturbance $\underline{w}_s$, and where $\underline{x}_d$ is the portion of the response due to $\underline{f}_d$. Let $\underline{y}_s$, $\underline{y}_d$, $\underline{z}_s$, $\underline{z}_d$, $\underline{Z}_s$, $\underline{Z}_d$, $\underline{u}_s$, and $\underline{u}_d$ be correspondingly defined.

$$\text{Then } J = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left\{ \mathscr{E}\left\langle [\underline{x}'_s + \underline{x}'_d] \begin{bmatrix} W_1 & W_2 \\ W'_2 & W_3 \end{bmatrix} \begin{Bmatrix} \underline{x}_s + \underline{x}_d \\ \underline{u}_s + \underline{u}_d \end{Bmatrix} \right\rangle \right\} dt \tag{3a}$$

can be reduced to $J = J_s + J_d$, where

$$J_s = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left\{ \mathscr{E}\left\langle [\underline{x}'_s + \underline{u}'_s] \begin{bmatrix} W_1 & W_2 \\ W'_2 & W_3 \end{bmatrix} \begin{Bmatrix} \underline{x}_s \\ \underline{u}_s \end{Bmatrix} \right\rangle \right\} dt \tag{3b}$$

and

$$J_d = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left\langle [\underline{x}'_d + \underline{u}'_d] \begin{bmatrix} W_1 & W_2 \\ W'_2 & W_3 \end{bmatrix} \begin{Bmatrix} \underline{x}_d \\ \underline{u}_d \end{Bmatrix} \right\rangle dt \tag{3c}$$

The problem is now separable into a stochastic– and a deterministic subproblem, each of which has an analytical solution. The two subproblems are stated, and their solutions presented (without development) below.

Stochastic Subproblem and Solution

Statement:

Given:

$$\dot{\underline{x}}_s = A\,\underline{x}_s + B\,\underline{u}_s + E_s\,\underline{w}_s \tag{4a}$$

$$\underline{y}_s = C\,\underline{x}_s + D\,\underline{u}_s \quad (\text{rank } C \le \dim \underline{x}_s) \tag{4b}$$

$$\underline{z}_s = \underline{y}_s + M\underline{n} \tag{4c}$$

{A,B} is stabilizable, {C,A} is detectable
$\underline{x}_s(0) = \underline{x}_{s0}$ is Gaussian with zero mean

$\underline{x}_{s0}$, $\underline{w}_s$, and $\underline{n}$ are independent and bounded

such that $\text{cov}[\underline{w}_s(t), \underline{w}_s(\tau)] = V_1\delta(t-\tau)$ (4d)

and $\text{cov}[\underline{n}(t), \underline{n}(\tau)] = V_3\delta(t-\tau)$ (4e)

where V_1 is PSD and V_3 is PD

$$J_s = \lim_{T\to\infty} \frac{1}{T} \left\{ \mathscr{E} \int_0^T \left\langle [\underline{x}_s' \ \underline{u}_s'] \begin{bmatrix} W_1 & W_2 \\ W_2' & W_3 \end{bmatrix} \begin{Bmatrix} \underline{x}_s \\ \underline{u}_s \end{Bmatrix} \right\rangle dt \right\} \quad (4f)$$

where W_1 is PSD and W_3 is PD

$\underline{Z}_s(t) = \{\underline{z}_s(\tau), 0 \le \tau \le t\}$, $\underline{u}_s(t) = \underline{\alpha}_s[t, \underline{Z}_s(t)]$ (4g)

defines the set of admissible controls

Find: An admissible control function $\underline{u}_s^*(t)$ which minimizes J_s with respect to the admissible control functions $\underline{u}_s(t)$

Solution (See ref. 6, pp. 272–277; and ref. 7, ch. 11):

$$\underline{u}_s^* (t) = -K \, \tilde{\underline{x}}_s(t) \quad (5a)$$

where $\tilde{\underline{x}}_s$ is an estimate of $\underline{x}_s$ using a Luenberger observer (ref. 7, pp. 288–289) having observer gain matrix L

$$K = W_3^{-1} (B'P + W_2') \quad (5b)$$

P is the unique PD solution to

$$PA + A'P - (PB + W_2) W_3^{-1}(PB + W_2)' + W_1 = 0 \quad (5c)$$

$$L = QC' (M V_3 M')^{-1} \quad (5d)$$

Q is the unique PD solution to

$$AQ + QA' - QC' (M V_3 M')^{-1} CQ + E_s V_1 E_s' = 0 \quad (5e)$$

P exists if $\{A,B\}$ is stabilizable and $\{C,A\}$ is detectable
or if the system is asymptotically stable

Q exists if $\{A, E_s V_1^{1/2}\}$ is stabilizable and $\{C,A\}$ is detectable
or if the system is asymptotically stable

Deterministic Subproblem and Solution

Statement:

Given:

$$\dot{\underline{x}}_d = A \underline{x}_d + B \underline{u}_d + E_d \underline{f}_d \quad (6a)$$

$$\underline{y}_d = C \underline{x}_d + D \underline{u}_d \quad (\text{rank } C \le \dim \underline{x}_d) \quad (6b)$$

$$\underline{z}_d = \underline{y}_d \quad (6c)$$

$\{A,B\}$ is stabilizable, $\{C,A\}$ is detectable

$\underline{x}_d(0) = \underline{x}_{d0}$

$\underline{x}_{d0}$ and $\underline{f}_d$ are independent and bounded

$$J_d = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left\langle [\underline{x}_d' \ \underline{u}_d'] \begin{bmatrix} W_1 & W_2 \\ W_2' & W_3 \end{bmatrix} \left\{ \begin{matrix} \underline{x}_d \\ \underline{u}_d \end{matrix} \right\} \right\rangle dt \qquad (6d)$$

where W_1 is PSD and W_3 is PD

$$\underline{Z}_d = \{\underline{z}_d(\tau), 0 \le \tau \le t\}, \ \underline{u}_d(t) = \underline{\alpha}_d[t, \underline{Z}_d(t), \underline{f}_d] \qquad (6e)$$

defines the set of admissible controls

Find: An admissible control function $\underline{u}_d{}^*(t)$ which minimizes J_d with respect to the set of admissible control functions $\underline{u}_d(t)$

Solution (refs. 8; 9; and 10, pp. 156–157):

$$\underline{u}_d{}^*(t) = -K\, \underline{x}_d - W_3^{-1} B' \int_t^{\infty} \exp[-\overline{A}'(t-\tau)]\, PE_d\, \underline{f}_d(\tau)\, d\tau \qquad (7a)$$

where $K = W_3^{-1} (B'P + W_2')$ (7b)

P is the unique PD solution to

$$PA + A'P - (PB + W_2)\, W_3^{-1} (PB + W_2)' + W_1 = 0 \qquad (7c)$$

P exists if {A,B} is stabilizable and {C,A} is detectable
or if the system is asymptotically stable

Combined Solution to Basic Problem

When rank C < dim $\underline{x}_d$, an estimate $\tilde{\underline{x}}_d$ of $\underline{x}_d$ must be used in the feedback. If one uses an asymptotic (i.e., Luenberger) observer, with gains L chosen to give an optimal solution to the stochastic subproblem, he can then combine the stochastic and deterministic subproblem solutions so as to use the same observer and regulator. This allows the optimal solution (feedback portion) to be realized physically. If such a choice is made,

$$\underline{u}^*(t) = u_d^*(t) + \underline{u}_d^*(t) = -K\, \tilde{\underline{x}}(t) - W_3^{-1} B' \int_t^{\infty} \exp[-\overline{A}'(t-\tau)]\, PE_d\, \underline{f}_d(\tau) d\tau \qquad (8a)$$

where $\tilde{\underline{x}}$ is an estimate of $\underline{x}$ using a Luenberger observer
having observer gain matrix L

$$K = W_3^{-1} (B'P + W_2') \qquad (8b)$$

$$L = QC'(M\, V_3\, M')^{-1} \qquad (8c)$$

P,Q, and $\overline{A}$ are as defined previously

If $\underline{f}_s$ and $\underline{n}$ are correlated by $\mathscr{E}\, [\underline{f}_s(t), \underline{n}(\tau)] = V_2\delta(t-\tau)$, then the above solution has the modification (ref. 7, pp. 414–417) that

$$L = (QC' + E_s V_2)(M\, V_3 M')^{-1} \qquad (8d)$$

where Q is the unique PD solution to

$$\tilde{A}Q + Q\tilde{A}' - QC'(M\, V_3 M')^{-1} CQ + E_s \tilde{V}_1 E_s' = 0 \qquad (8e)$$

$$\text{for } \tilde{A} = A - E_s V_2 V_3^{-1} C \tag{8f}$$
$$\text{and } \tilde{V}_1 = V_1 - V_2 V_3^{-1} V_2' \tag{8g}$$

PROBLEM EXTENSIONS

Frequency Weighting

Suppose now that it is desired to frequency weight the states $\underline{x}$ and the control $\underline{u}$ in the cost rate functional, so that the weightings vary with frequency (ref. 11). Let $\underline{x}$ be considered to be the input to a filter $\mathscr{W}_1(s)$ of which ${}^1\underline{x}$ is the output, and let $\mathscr{W}_1(s)$ have a state–space representation defined by $\{A_1,B_1,C_1,D_1\}$ [i.e., $\mathscr{W}_1(s) = C_1(sI-A_1)^{-1}B_1+ D_1$]. Then

$$\dot{\underline{z}}_1 = A_1\,\underline{z}_1 + B_1\,\underline{x} \tag{9a}$$
$$ {}^1\underline{x} = C_1\,\underline{z}_1 + D_1\,\underline{x} \tag{9b}$$

expresses ${}^1\underline{x}$ in terms of $\underline{x}$, employing pseudostates $\underline{z}_1$. Similarly, if $\underline{u}$ is considered to be the input to a filter $\mathscr{W}_3(s)$ of which ${}^1\underline{u}$ is the output, and if $\mathscr{W}_3(s)$ has a state–space representation defined by $\{A_2,B_2,C_2,D_2\}$, ${}^1\underline{u}$ can be expressed in terms of $\underline{u}$, employing pseudostates $\underline{z}_2$:

$$\dot{\underline{z}}_2 = A_2\,\underline{z}_2 + B_2\,\underline{u} \tag{10a}$$
$$ {}^1\underline{u} = C_2\,\underline{z}_2 + D_2\,\underline{u}_2 \tag{10b}$$

Suppose now that these frequency–weighted states (${}^1\underline{x}$) and controls (${}^1\underline{u}$) are further weighted by constant weighting matrices W_1 and W_2, respectively. The resulting state equations and performance index are as follows:

$$\dot{\bar{\underline{x}}} = {}^1A\,\bar{\underline{x}} + {}^1B\,\underline{u} + {}^1E_d\underline{f}_d + {}^1E_s\,\underline{w}_s \tag{11a}$$

$$\underline{y} = {}^1C\,\bar{\underline{x}} + D\,\underline{u} \tag{11b}$$

$$\underline{z} = \underline{y} + M\,\underline{n} \tag{11c}$$

$$ {}^1J = \mathscr{E}\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\langle[\bar{\underline{x}}'\ \underline{u}']\begin{bmatrix}{}^1W_1 & {}^1W_2\\ {}^1W_2' & {}^1W_3\end{bmatrix}\left\{\begin{matrix}\bar{\underline{x}}\\ \underline{u}\end{matrix}\right\}\right\rangle dt\right\} \tag{11d}$$

where
$$\bar{\underline{x}} = \begin{bmatrix}\underline{x}\\ \underline{z}_1\\ \underline{z}_2\end{bmatrix} \tag{11e}$$

$$ {}^1A = \begin{bmatrix}A & O & O\\ B_1 & A_1 & O\\ O & O & A_2\end{bmatrix} \tag{11f}$$

$$^{1}B = \begin{bmatrix} B \\ O \\ B_2 \end{bmatrix} \tag{11g}$$

$$^{1}C = [C \; O \; O] \tag{11h}$$

$$^{1}E_d = \begin{bmatrix} E_d \\ O \\ O \end{bmatrix} \tag{11i}$$

$$^{1}E_s = \begin{bmatrix} E_s \\ O \\ O \end{bmatrix} \tag{11j}$$

$$^{1}W_1 = \begin{bmatrix} D_1'W_1D_1 & D_1'W_1C_1 & O \\ C_1'W_1D_1 & C_1'W_1C_1 & O \\ O & O & C_1'W_3C_2 \end{bmatrix} \tag{11k}$$

$$^{1}W_2 = \begin{bmatrix} O \\ O \\ C_2'W_3D_2 \end{bmatrix} \tag{11l}$$

$$^{1}W_3 = [D_2'W_3D_2] \tag{11m}$$

Disturbance Accommodation

Suppose further that the stochastic disturbance is not $\underline{w}_s$ but $\underline{f}_s$, where $\underline{f}_s$ is a stochastically modeled disturbance with power spectral density $S_f(\omega) = S_f^{1/2}(j\omega)S_f^{1/2^*}(j\omega)$. Defining $H_f(j\omega)$ by $S_f^{1/2}(j\omega)\, V_1^{1/2}$, one can consider $\underline{f}_s$ to be the output of a filter $H_f(s)$ excited by zero–mean white Gaussian noise $\underline{w}_s$ (ref. 12) with power V_1 (i.e., $\text{cov}[\underline{w}_s(t), \underline{w}_s(\tau)] = V_1\, \delta(t-\tau)$.
In state–space form,

$$\dot{\underline{\xi}} = A_s\, \underline{\xi} + \underline{w}_s \tag{12a}$$

$$\underline{f}_s = C_s(sI - A_s)^{-1} \tag{12b}$$

such that
$$H_f(s) = C_s(sI - A_s)^{-1} \tag{12c}$$

Incorporating these new pseudostates ($\underline{\xi}$) into the state equations and performance index yields

$$\dot{\hat{\underline{x}}} = {}^2A\,\hat{\underline{x}} + {}^2B\,\underline{u} + {}^2E_d\,\underline{f}_d + {}^2E_s\,\underline{w}_s \tag{13a}$$

$$\underline{y} = {}^2C\,\hat{\underline{x}} + D\,\underline{u} \tag{13b}$$

$$\underline{z} = \underline{y} + M\,\underline{n} \tag{13c}$$

$$ {}^2J = \mathscr{E}\left[\lim_{T\to\infty} \frac{1}{T} \int_0^T \left\langle [\hat{\underline{x}}'\ \underline{u}'] \begin{bmatrix} {}^1W_1 & {}^1W_2 \\ {}^1W_2' & {}^1W_3 \end{bmatrix} \left\{ \begin{matrix} \hat{\underline{x}} \\ \underline{u} \end{matrix} \right\} \right\rangle dt \right] \tag{13d}$$

where $$\hat{\underline{x}} = \left\{ \begin{matrix} \overline{x} \\ \underline{\xi} \end{matrix} \right\} \tag{13e}$$

$$ {}^2A = \begin{bmatrix} A & O & O & E_s C_s \\ B_1 & A_1 & O & O \\ O & O & A_2 & O \\ O & O & O & A_s \end{bmatrix} \tag{13f}$$

$$ {}^2B = \begin{bmatrix} B \\ O \\ B_2 \\ O \end{bmatrix} \tag{13g}$$

$$ {}^2C = [C\ O\ O\ O] \tag{13h}$$

$$ {}^2E_d \begin{bmatrix} E_d \\ O \\ O \\ O \end{bmatrix}$$

$$ {}^2E_s \begin{bmatrix} O \\ O \\ O \\ I \end{bmatrix} \tag{13j}$$

$$ {}^2W_1 = \begin{bmatrix} D_1'W_1D_1 & D_1'W_1C_1 & O & O \\ C_1'W_1D_1 & C_1'W_1C_1 & O & O \\ O & O & C_2'W_3C_2 & O \\ O & O & O & O \end{bmatrix} \tag{13k}$$

$$
{}^2W_2 = \begin{bmatrix} O \\ O \\ C_2'W_3D_2 \\ O \end{bmatrix} \tag{13m}
$$

$$
{}^2W_3 = [D_2'W_3D_2] \tag{13n}
$$

The solution to this problem has been given previously.

SYNTHESIS MODEL

The model given at the close of the previous section is the model from which the controller is synthesized. The synthesis involves the determination of observer gains L and regulator feedback gains K. Preview gains K_{FF} can also be determined, if desired, to approximate the Duhamel integral term of the optimal control. One approach to determining these preview gains has been presented in reference 9. Further study of the determination and use of these gains is needed.

ANALYSIS MODEL

Once the controller has been selected, it must be connected to the actual plant and the resulting "analysis model" used to evaluate closed–loop–system performance and stability. For constant gain matrices K, L, and K_{FF} the open loop transfer function from $\underline{Y}$ to $\underline{U}_{FB}$ [$=-K\,\underline{\hat{X}}$] is

$$
\mathscr{H}^{OL}_{\underline{U}_{FB}\underline{Y}}(s) = \left[\begin{array}{c|c} {}^2A-{}^2BK-L\,{}^2C & L \\ \hline -K & O \end{array}\right]; \tag{14a}
$$

where the form $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

indicates $C(sI-A)^{-1}B+D$. The closed loop transfer functions, respectively, from $\underline{F}_d$ and $\underline{F}_s$ to $\underline{X}$, are

$$
\mathscr{H}^{CL}_{\underline{X}\,\underline{F}_d}(s) = \left[\begin{array}{cc|c} A & -BK & E_d+BK_{FF} \\ LC & {}^2A-{}^2BK-L\,{}^2C & {}^2BK_{FF} \\ \hline I & O & O \end{array}\right] \tag{14b}
$$

$$
\text{and } \mathscr{H}^{CL}_{\underline{X}\,\underline{F}_s}(s) = \left[\begin{array}{cc|c} A & -BK & E_s+BK_{FF} \\ LC & {}^2A-{}^2BK-L\,{}^2C & {}^2BK_{FF} \\ \hline I & O & O \end{array}\right] \tag{14c}
$$

The return ratio matrices (ref. 4) at the $\underline{Y}(s)$ and $\underline{U}(s)$ nodes, respectively, for $D \equiv O$, are

$$L_2(s) = \left[\begin{array}{cc|c} A^2 - BK - L^2C & 0 & L \\ -BK & A & O \\ \hline O & -C & O \end{array}\right] \qquad (15a)$$

$$L_1(s) = \left[\begin{array}{cc|c} {}^2A - {}^2BK - L^2C & LC & O \\ O & A & B \\ \hline K & O & O \end{array}\right] \qquad (15b)$$

The corresponding return difference matrices and inverse return difference matrices (ref. 4) are as follows:

$$I + L_2(s) = \left[\begin{array}{cc|c} {}^2A - {}^2BK - L^2C & O & L \\ -BK & A & O \\ \hline O & -C & I \end{array}\right] \qquad (15c)$$

$$I + L_1(s) = \left[\begin{array}{cc|c} {}^2A - {}^2BK - L^2C & LC & O \\ O & A & B \\ \hline K & O & I \end{array}\right] \qquad (15d)$$

$$I + L_2^{-1}(s) = I + [K(sI - {}^2A + {}^2B\,K + L^2C)^{-1}L]^{-1}[C(sI - A)^{-1}B]^{-1} \qquad (15e)$$

$$I + L_1^{-1}(s) = I + [C(sI - A)^{-1}B]^{-1}[K(sI - {}^2A + {}^2B\,K + L^2C)^{-1}L]^{-1} \qquad (15f)$$

The singular values of these matrices can be used to evaluate system noise and disturbance attenuation, stability margins, and sensitivity (ref. 4). Iterative application of the synthesis– and analysis models can be used to produce the desired controller.

EXAMPLE PROBLEM

Suppose one wishes to develop a controller to isolate a space experiment of mass m and position x(t), from a unidirectional acceleration disturbance $\ddot{d}(t)$. Assume that a wall having position d(t) acts on m through an umbilical with stiffness k and damping c. (See figure 1). Suppose further that rattlespace constraints require the transmissibility to be unity below 10^{-3} Hz, and that it is desired to attenuate the disturbance by at least two orders of magnitude between 0.05 and 10 Hz. Let a linear actuator, applying a force that varies with control current i, be connected between the wall and the experiment in parallel with the umbilical.

For this problem, it is desirable at low frequencies to penalize the relative displacement of the experiment heavily, so that the experiment "tracks" the wall. At intermediate frequencies, however, the absolute acceleration of the experiment should be heavily penalized to accomplish the desired disturbance rejection. The state space model, then, should have relative position x–d and absolute acceleration $\ddot{x}$ as states, allowing them to be frequency–weighted in the performance index.

The system equation of motion is

$$\ddot{x} = -\hat{k}(x-d)-\hat{c}(\dot{x}-\dot{d}) - \hat{\alpha}i, \text{ where } \hat{k} = \frac{k}{m}, \hat{c} = \frac{c}{m}, \text{ and } \hat{\alpha} = \frac{\alpha}{m} \tag{16a}$$

In state–space form, the equations can be written as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\hat{k} & -\hat{c} & 0 \\ -\omega_h\hat{k} & -\omega_h\hat{c} & -\omega_h \end{bmatrix} + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \begin{Bmatrix} 0 \\ -\hat{\alpha} \\ -\omega_h\hat{\alpha} \end{Bmatrix} i + \begin{Bmatrix} 0 \\ -1 \\ 0 \end{Bmatrix} \ddot{d} \tag{16b}$$

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \tag{16c}$$

$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} \tag{16d}$$

where

$$\begin{aligned} x_1(t) &= x(t) - d(t) \\ x_2(t) &= \dot{x}(t) - \dot{d}(t) \\ X_3(s) &= \left(\frac{\omega_h}{s+\omega_h}\right) s^2X(s), \ \omega_h \text{ high,} \end{aligned} \tag{16e}$$

so that $x_3(t) \approx \ddot{x}(t)$ for $\omega << \omega_h$

Frequency–weighting the states so that

$$\begin{Bmatrix} {}^1X_1(s) \\ {}^1X_2(s) \\ {}^1X_3(s) \end{Bmatrix} = \begin{bmatrix} \frac{\omega_3}{s} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & \frac{\omega_2 s}{(s+\omega_1)(s+\omega_2)} \end{bmatrix} \begin{Bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{Bmatrix} \tag{17a}$$

(where $\omega_1 < \omega_2$) results in a performance index that penalizes x_1 more highly at low frequencies and x_3 more highly at intermediate frequencies. If the control is frequency–weighted so that

$${}^1U(s) = \left(\frac{\omega_4 s}{s+\omega_4}\right) U(s) \ [\omega_4 < \omega_h] \ , \tag{17b}$$

at higher frequencies the control will be more heavily penalized. This is desirable both for the sake of robustness and since x_3 approximates $\ddot{x}$ only at frequencies sufficiently below ω_h. Finally, let the input acceleration be considered to come from zero–mean Gaussian white noise filtered through $\frac{\omega_f}{s+\omega_f}$.

The resultant state equations are as indicated on page 8, where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(\omega_1+\omega_2) & -\omega_1\omega_2 \\ 0 & 1 & 0 \end{bmatrix} \tag{18a}$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{18b}$$

$$C_1 = \begin{bmatrix} \omega_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_2 & 0 \end{bmatrix} \tag{18c}$$

$$D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{18d}$$

$$A_2 = -\omega_4$$
$$B_2 = 1$$
$$C_2 = -\omega_4^2$$
$$D_2 = \omega_4$$
$$A_s = -\omega_f$$
$$B_s = 1$$
$$C_s = \omega_f$$
$$D_s = 0$$

Assume that $\operatorname{cov}[w_s(t), w_2(\tau)] = 1\ \delta(t-\tau)$

$\operatorname{cov}[n_1(t), n_1(\tau)] = 0.001\ \delta(t-\tau)$

and $\operatorname{cov}[n_2(t), n_2(\tau)] = 0.001\ \delta(t-\tau)$.

Since A_1 has a zero [1st] column, ${}^2A$ will have a corresponding zero [4th] column. To make the frequency–weighted system $\{{}^2C, {}^2A\}$ observable, obtain $\int (x-d)\,dt$ as a measured state (i.e., the first pseudostate, Z_{11}) and modify ${}^2C$ accordingly. Let the measurement noise associated with Z_{11} be n_3, such that

$$\operatorname{cov}[n_3(t), n_3(\tau)] = 0.0001$$

Gain matrix W_1 can be varied to "tune" the optimal control to give the most satisfactory results. The transmissiblity between $\ddot{d}(t)$ and $\ddot{x}(t)$ is given in figure 2. The control uses feedback (and observer) gains obtained from system parameters and weightings as indicated on the figure. Note that the low–frequency transmissibility is unity, as desired, and that for intermediate frequencies the transmissibility rolls off with a slope of –1.

If a different frequency–weighting of x_3 is used, it is to be anticipated that the transmissibility curve will change as well.

For $${}^1X_3(s) = \frac{\omega_2^2\, s}{(s+\omega_1)^2(s+\omega_2)^2} X_3(s) \qquad (19)$$

the resultant selected transmissibility curve is given in figure 3. The low–frequency transmissibility again, is unity; but now for the intermediate frequencies the transmissibility rolls off with a slope of –2, as expected. Adding another pole at ω_1 and at ω_2 to the $X_3(s)$ frequency weighting would further improve the intermediate–frequency roll–off. The present controller, however, meets the design specifications.

If state frequency–weightings of $${}^1X_1(s) = \frac{\omega_3}{s+\omega_3} X_1(s) \qquad (20a)$$

and $${}^1X_3(s) = \frac{\omega_2^2\, s}{(s+\omega_1)^2(s+\omega_2)^2} X_3(s) \qquad (20b)$$

are used, the results (figure 4) are similar to those given previously in figure 3. Note that with this latter choice of frequency weighting however, (i.e., without any "rigid body poles"), the frequency–weighted system $\{{}^2C, {}^2A\}$ is observable, without augmenting the actual plant output $\underline{y}$ as was previously necessary. Consequently this is the preferred control.

DISCUSSION

H_2 synthesis, as the example problem indicates, provides a highly versatile loop–shaping tool. It is especially useful in controller development for SIMO and MIMO systems, where classical loop–shaping methods are most lacking. Once the designer has expressed the system equations in terms of states for which he has an intuitive feel, and of measurable outputs, the design process becomes relatively easy. He frequency weights (i.e., filters) the states and control inputs according to his engineering experience and intuition, to indicate the relative importance of each as a function of frequency. Then he weights these frequency–weighted states and controls relative to each other. The H_2 synthesis methodology automatically provides him with a set of regulator and observer gains that are optimal with respect to the chosen weightings, given a quadratic performance index. Known aspects of the input disturbances and sensor noise can be incorporated into the design as well. Singular value checks provide the ability to evaluate system robustness. With a few iterations, the skillful engineer can complete his design. Excellent computer software packages already exist to assist in the task.

The frequency weighting tells the H_2 synthesis machinery how much "cost" to place on a state or control input at any frequency, relative to its cost at other frequencies. If, for example, absolute acceleration is undesirable only in a particular frequency range, that is where it should be most heavily weighted. The subsequent weighting of the frequency–weighted states and control inputs tell the synthesis machinery how much cost to place on each frequency–weighted state or control relative to the others.

In the example problem changing the relative weighting between absolute acceleration and relative displacement caused the frequency range of unit transmissibility to vary. An increase in the sharpness of the bandpass filter, used in the acceleration frequency weighting, resulted in a corresponding increase in the rate of gain roll–off. Increasing the weighting of relative velocity added damping to the system, as expected; and adjusting the acceleration bandpass filter's lower pole location allowed fine tuning of the unit transmissibility upper frequency limit. Use of a high–pass filter for control weighting produced a control which responds favorably (i.e., minimally) at higher frequencies, where the plant models typically are invalid.

CONCLUDING REMARKS

The extended H_2 synthesis method has been developed and applied to a one–dimensional microgravity vibration isolation problem, for which it seems particularly well–suited. Research continues toward the application of H_2 synthesis to the full six–degree–of–freedom isolation problem.

SYMBOLS AND ABBREVIATIONS

Alphabctical Symbols

A	System dynamic matrix
B	System control input matrix
c	Umbilical damping
C	Systems State output matrix
D	Control transmission matrix
E	System disturbance input matrix
$\underline{f}$	Disturbance vector
H	Stochastic–disturbance input filter
$\mathscr{H}(s)$	Transfer function matrix
i	Control current
I	Identy matrix
j	Square root of –1
J	Performance index
K	Control feedback gain matrix
L	Observer gain matrix
m	Experiment mass
M	Sensor noise input matrix
$\underline{n}$	Sensor noise vector
O	Zero matrix
P	Algebraic Riccati Equation solution for regulator feedback gains
Q	Algebraic Riccati Equation solution for observer gains
s	Laplace variable
S(s)	Stochastic–disturbance power–spectral–density matrix
t	Time
$\underline{u}$	Control vector
V	Covariance matrix
$\underline{w}$	White–noise disturbance vector
W, $\mathscr{W}$	Weighting matrix
α	Actuator proportionality constant
$\underline{\alpha}$	Admissible–control function
δ	Dirac delta function
$\mathscr{E}$	Expected–value operator
$\underline{\xi}$	Disturbance–accommodation pseudostates
ω	Circular frequency
Capitalization	Laplace transform, indicated by context

Abbreviations

CL	Closed loop
cov	Covariance
FF	Feedforward (preview) gain
MDOF	Multiple–degree–of–freedom
MIMO	Multiple–input–multiple–output
OL	Open loop
PD	Positive definite
PSD	Positive semidefinite
SDOF	Single–degree–of–freedom
SIMO	Single–input–multiple–output

Subscripts, Superscripts, and Diacritical Marks

postsubscript 0	Value at time t=0
postsubscript 1	With A,B,C,D,$\underline{z}$: related to state–frequency weighting state–space description With L: return ratio matrix at control node With V: process noise covariance With w,W,$\mathscr{W}$: state (or pseudostate) weightings, applied subsequent to any frequency weighting
postsubscript 2	With A,B,C,D,$\underline{z}$: related to state–frequency–weighting state–space description With L: return ratio matrix at output node. With V,W: cross–weightings
postsubscript 3	With V: measurement noise covariance With W,$\mathscr{W}$: control weightings
postsubscript d	Related to deterministic disturbance
postsubscript f	Related to filter for stochastic disturbance
postsubscript s	Related to stochastic disturbance
postsuperscript 1/2	Square root or spectral factorization
postsuperscript '	Transpose
postsuperscript –1	Inverse
postsuperscript *	Optimum or conjugate transpose
underline _	Vector
overbar ¯	With A: closed loop system dynamic matrix With $\bar{x}$: augmented with frequency–weighting pseudostates
overhat ˆ	Augmented with frequency–weighting– and disturbance–accommodation pseudostates
overtilde ~	Estimated or associated with cross–correlation
presuperscript 1	With $\underline{x}$,$\underline{X}$,$\underline{u}$, or $\underline{U}$: frequency–weighted With other symbols: related to system augmented by frequency weighting
presuperscript 2	Related to system augmented by frequency–weighting and disturbance–accommodation

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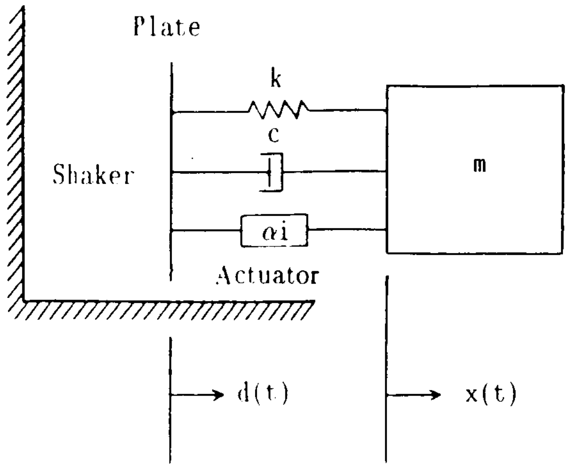

m = 75 lbm
k = 1.544 lbf/ft
c = 0 lbf-sec/ft
α = 2 lbf/amp

FIGURE 1: Example-Problem Model

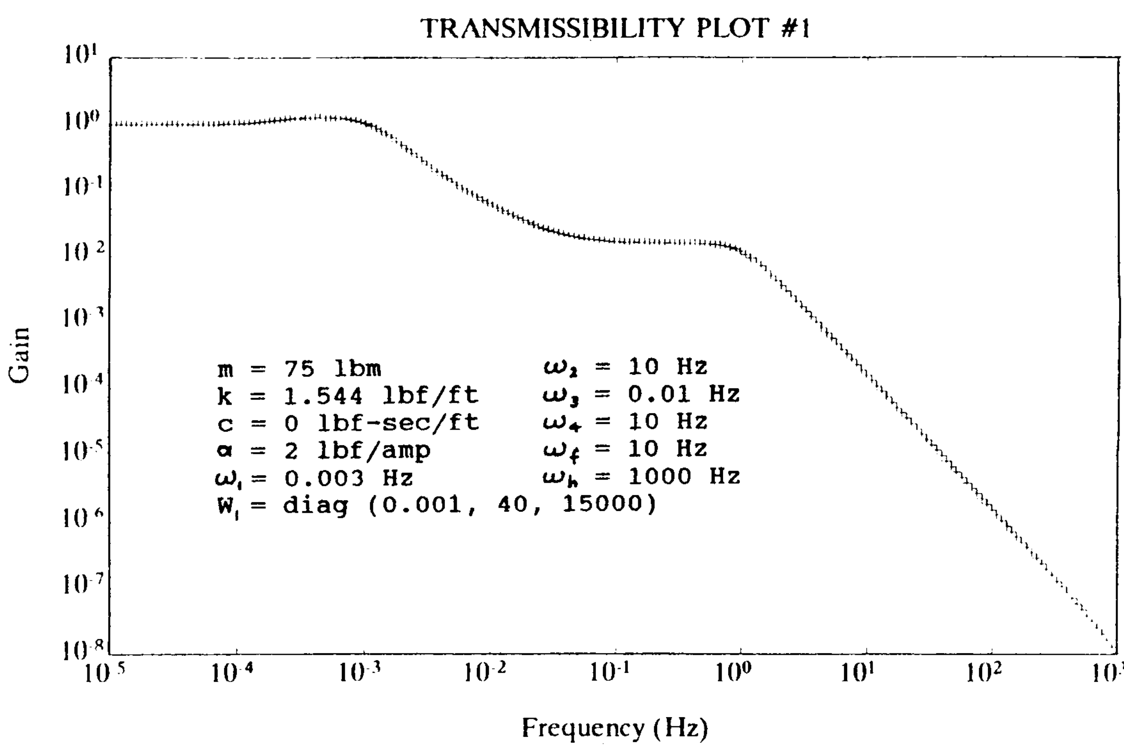

FIGURE 2: Transmissibility Plot for 1st Control Weighting

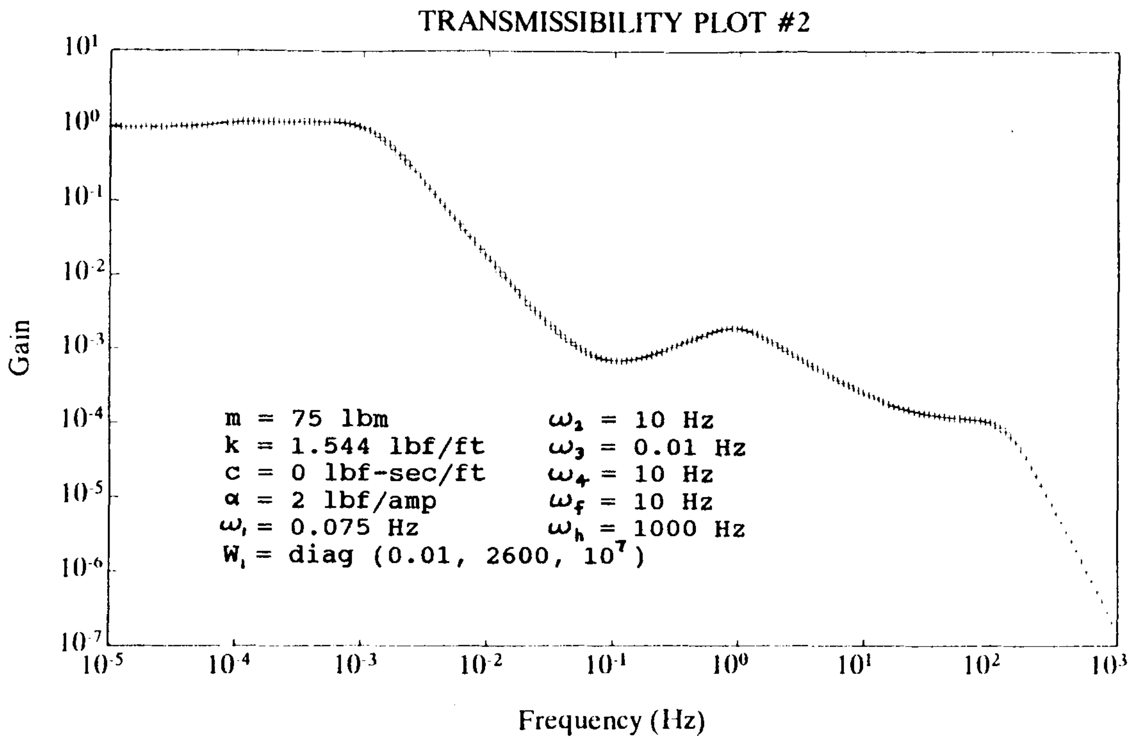

FIGURE 3: Transmissibility Plot for 2nd Control Weighting

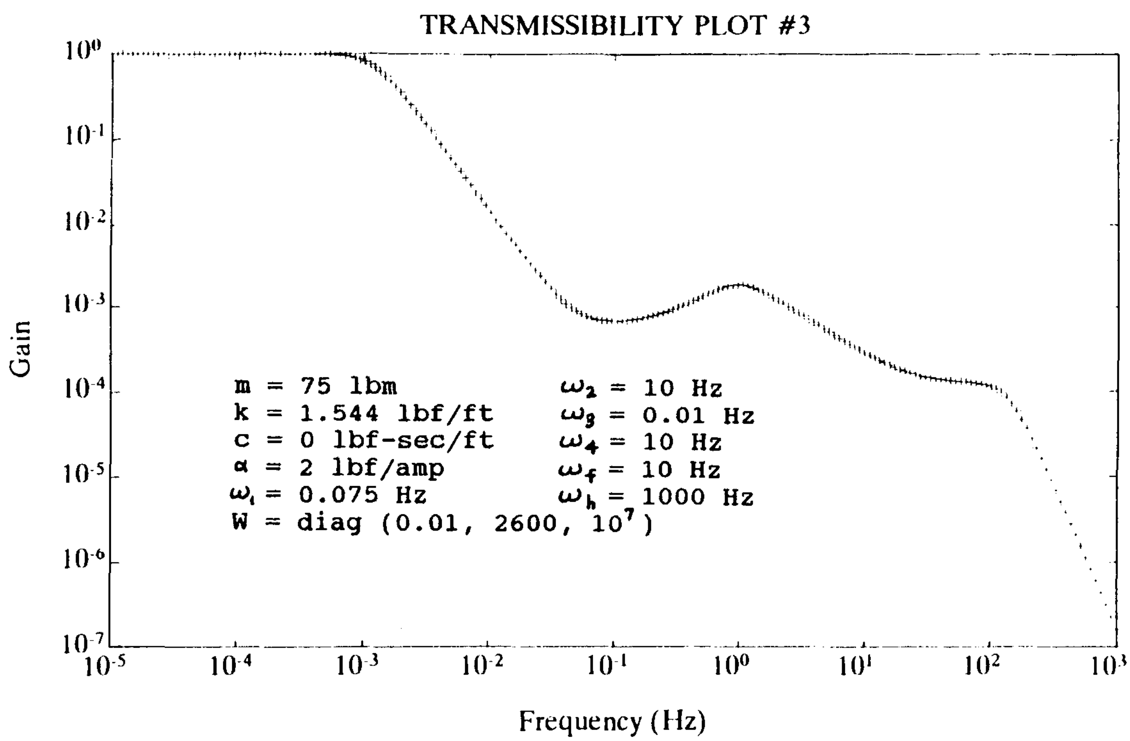

FIGURE 4: Transmissibility Plot for 3rd Control Weighting