DYNAMIC MODEL OF A 5-AXIS MAGNETIC BEARING SYSTEM WITH FLEXIBLE ROTOR

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ABSTRACT

This paper describes the derivation of a nonlinear ordinary differential equation model for a flexible rotor supported by magnetic bearings. The model is based on a finite element formulation which divides the rotor into flexible Euler-Bernoulli beam elements. In addition to the Euler-Bernoulli assumption, the derivation assumes no axial deformation, no internal damping, and no mass unbalance. The derived model can be readily implemented in Matlab for purposes of time-domain simulation.

INTRODUCTION

There is considerable interest in using active magnetic bearings to control undesirable vibrations in flexible shafts. Magnetic bearings are useful for providing vibration control as their stiffness and damping can be adjusted on-line. A good introduction to some of the challenges in flexible rotor modelling and control are described in [9]. Many control schemes which account for rotor flexibility are based on Linear Time Invariant (LTI) models which usually result from some form spatial discretization such as the Finite Element Method (FEM) [7, 6] followed possibly by a model-order reduction [5, 11, 2]. Much of the modelling work involving magnetic bearings and flexible rotors makes use of results from conventional bearing-rotordynamic literature e.g. [7]. An LTI model format is a convenient starting point for the application of many robust control design methods such as H_∞-optimal control [8, 11, 1, 4]. Other work on lower dimensional modelling of flexible rotors supported by magnetic bearings makes use of a nonlinear Jeffcott model [3].

This paper will consider the derivation of a FEM model which which is nonlinear. The model derivation uses the framework of Shabana [10] and generalizes the rigid body model derived in [14, 12, 13] to which flatness-based nonlinear control methods have been successfully applied. It is expected that by extending this rigid body model to account for flexibility, nonlinear control strategies such as those described in [14, 12, 13] could be successfully applied.

*This work was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC), under grant number 249681-02.

FLEXIBLE SHAFT MODEL

We consider a shaft supported by one axial and two radial magnetic bearings as shown schematically in Fig. 1. The shaft is rotated by an electric motor. The axial bearing force is denoted F_x, the radial forces at the front of the shaft are denoted $F_{v,y}, F_{v,z}$, and the radial forces at the rear of the shaft are denoted $F_{h,y}, F_{h,z}$. We take the shaft's length to be L, its mass as m, and assume that the shaft's cross-sectional area is small relative to its length. As well, for simplicity we ignore shear deformation, internal damping, and deformation in the axial direction. We divide the shaft into N flexible beam elements and take l_i, m_i, ρ_i, EI_i to be the length, mass, volumetric density, and flexural rigidity of element i respectively ($1 \leq i \leq N$). The following coordinate frames are defined: $OXYZ$ is an inertial frame whose origin is fixed and located at the centre of the shaft's front face when the shaft is un-deformed and centred between all bearings. The direction of the X, Y, and Z axes are shown in Figure 1 with the Z axis pointing out the page. The $oxyz$ floating frame is rigidly attached to the centre of the front face of the shaft. The origin of $oxyz$ is displaced from $OXYZ$ by a vector R. The orientation of the $oxyz$ frame relative to $OXYZ$ is described by three angles ψ, θ, and ϕ. The $OXYZ$ frame is rotated by an angle ψ about its Y-axis. Call this rotated frame $OX'Y'Z'$. The $OX'Y'Z'$ frame is then rotated an angle θ about its Z' axis. Call this rotated frame $OX''Y''Z''$. Finally, the orientation of $oxyz$ is obtained by a rotation of ϕ about the X'' axis. From [14], the ro-

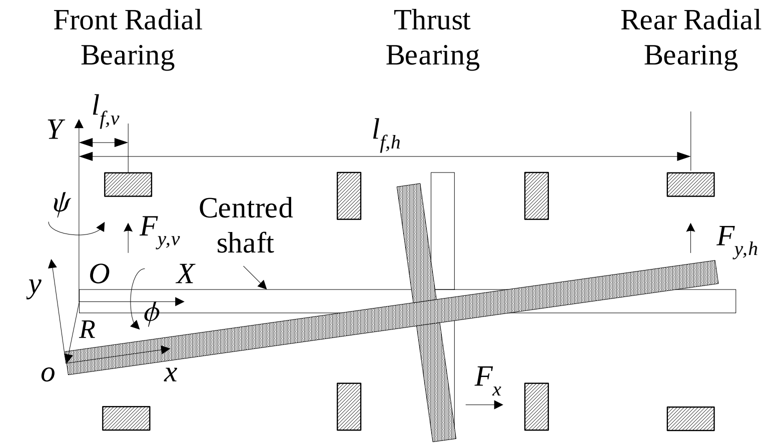

FIGURE 1: Un-deformed shaft supported by magnetic bearings. Z, z-axes point out of the page.

tation matrix which maps vectors represented in $oxyz$ to $OXYZ$ is given by

$$A = \begin{bmatrix} c\psi c\theta & s\psi s\phi - c\psi s\theta c\phi & c\psi s\theta s\phi + s\psi c\phi \\ s\theta & c\theta c\phi & -c\theta s\phi \\ -s\psi c\theta & s\psi s\theta c\phi + c\psi s\phi & c\psi c\phi - s\psi s\theta s\phi \end{bmatrix}$$

where $c\phi = \cos\psi, s\phi = \sin\phi$. An element frame $O_iX_iY_iZ_i$ is rigidly attached to element i and its origin is taken at the centre of the front face of element i, the X_i-axis remains co-linear with the un-deformed centreline of the element, see Figure 2. The front, respectively rear, radial actuators are located at distances $l_{f,v}$, respectively $l_{f,h}$, in the X-direction from the origin of the $OXYZ$ frame.

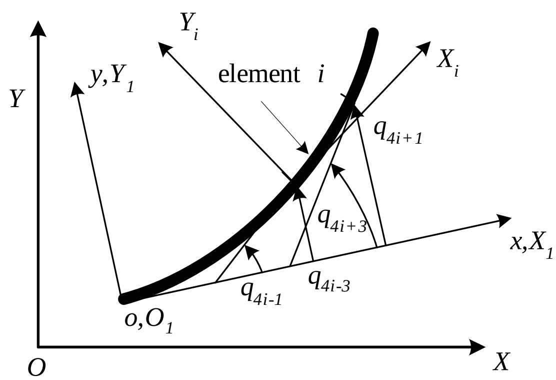

FIGURE 2: Projection of the shaft onto the xy-plane and nodal coordinates of element i.

We define nodal coordinates for the beam elements using projections of the shaft onto the xy and xz-planes. Projecting the shaft onto the xy-plane, respectively the xz-plane, we denote q_{4i-3} and q_{4i-2} as the distance of the point of intersection of elements $i-1$ and i to the x-axis. We denote q_{4i-1}, respectively q_{4i}, as the flexural slope of the shaft projected onto the xy-plane, respectively xz-plane, at the point of intersection of elements $i-1$ and i. These slopes are measured with respect to x-axis. The coordinates q_{4N+1}, q_{4N+2} denote the flexural displacements at the rear tip of the shaft, and q_{4N+3}, q_{4N+4} denote the flexural slopes at the rear tip of the shaft. Because the origin of the floating frame is defined at the center of the front face of the shaft, the first four nodal coordinates in the first element are zero.

The shape function of the ith element is

$$\begin{aligned} S^i &= \begin{bmatrix} S_1 & 0 & S_2 & 0 & S_3 & 0 & S_4 & 0 \\ 0 & S_1 & 0 & S_2 & 0 & S_3 & 0 & S_4 \end{bmatrix} \\ &= \begin{bmatrix} S_2^i(x_i) \\ S_3^i(x_i) \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} S_1(\xi_i) &= 1 - 3\xi_i^2 + 2\xi_i^3, & S_2(\xi_i) &= l_i(\xi_i - 2\xi_i^2 + \xi_i^3) \\ S_3(\xi_i) &= 3\xi_i^2 - 2\xi_i^3, & S_4(\xi_i) &= l_i(\xi_i^3 - \xi_i^2) \end{aligned}$$

$\xi_i = x_i/l_i$ is a normalized variable, and x_i is a displacement in the X_i-direction. The position of a point in the ith element with un-deformed coordinates (x_i, y_i, z_i) in the $O_iX_iY_iZ_i$ frame can be expressed in the floating frame $oxyz$ as

$$\begin{aligned} r^i &= \begin{bmatrix} \bar{r}^i(x_i) \\ u_i(x_i) + y_i \\ v_i(x_i) + z_i \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{i-1} l_k + x_i \\ S_2^i(x_i)q_f^i + y_i \\ S_3^i(x_i)q_f^i + z_i \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=1}^{i-1} l_k + x_i \\ S_2^i(x_i)B^i q_f + y_i \\ S_3^i(x_i)B^i q_f + z_i \end{bmatrix} \end{aligned}$$

when it experiences a deformation. Evidently, the expression for r^i is approximate and assumes small deformations. The components of q_f^i are the nodal coordinates of the ith element:

$$q_f^i = [q_{4i-3}\ q_{4i-2}\ q_{4i-1}\ q_{4i}\ q_{4i+1}\ q_{4i+2}\ q_{4i+3}\ q_{4i+4}]^T$$

and B^i is a Boolean index matrix used to assemble the elements

$$B^i = \begin{bmatrix} 0_{8\times 4(i-1)} & I_{8\times 8} & 0_{8\times 4(N-i)} \end{bmatrix}$$

From the definition of the shape function we have

$$\begin{aligned} u_i(0) &= q_{4i-3}, u_i'(0) = q_{4i-1}, \\ u_i(l_i) &= q_{4i+1}, u_i'(l_i) = q_{4i+3}, \\ v_i(0) &= q_{4i-2}, v_i'(0) = q_{4i}, \\ v_i(l_i) &= q_{4i+2}, v_i'(l_i) = q_{4i+4} \end{aligned}$$

Letting r_0^i be the position of a point mass in element i expressed in $OXYZ$

$$r_0^i = R + Ar^i \tag{1}$$

where R is the vector from the origin of $OXYZ$ to that of $oxyz$ expressed in $OXYZ$. Defining $N^i(x_i) = \begin{bmatrix} 0 \\ N_2^i(x_i) \\ N_3^i(x_i) \end{bmatrix} = \begin{bmatrix} 0 \\ S_2^i(x_i)B^i \\ S_3^i(x_i)B^i \end{bmatrix}$, we can rewrite r_0^i as

$$r_0^i = R + Ar^i = R + A \begin{bmatrix} \bar{r}^i \\ y_i \\ z_i \end{bmatrix} + AN^i(x_i)q_f$$

Taking the time derivative of r_0^i and denoting ω_{kf} as the velocity vector of frame $oxyz$ relative to $OXYZ$ expressed in $OXYZ$ gives

$$\begin{aligned} \dot{r}_0^i &= \dot{R}_0 + \dot{A}r^i + A\dot{r}^i = \dot{R}_0 + A(\omega_{kf} \times r^i) + AN^i\dot{q}_f \\ &= \dot{R}_0 - A(r^i \times \omega_{kf}) + AN^i\dot{q}_f \\ &= \dot{R}_0 - A\tilde{r}^i\omega_{kf} + AN^i\dot{q}_f \\ &= \dot{R}_0 - A\tilde{r}^i G\dot{\Phi} + AN^i\dot{q}_f \end{aligned}$$

where $\Phi = \begin{bmatrix} \phi & \psi & \theta \end{bmatrix}^T$,

$$G = \begin{bmatrix} 1 & s\theta & 0 \\ 0 & c\phi c\theta & s\phi \\ 0 & -s\phi c\theta & c\phi \end{bmatrix},$$

$$\tilde{r}^i = \begin{bmatrix} 0 & -(v^i + z_i) & u^i + y_i \\ v^i + z_i & 0 & -\bar{r}^i \\ -(u^i + y_i) & \bar{r}^i & 0 \end{bmatrix}$$

Letting $q = [X\ \ Y\ \ Z\ \ \phi\ \ \psi\ \ \theta\ \ q_5\ \ q_6 \ldots\ \ q_{4N+4}]^T = [q_r^T\ \ q_f^T]^T$, we express the kinetic energy of element i:

$$T^i = \frac{1}{2}\int_{V^i} \rho_i \dot{r}_0^{iT}\dot{r}_0^i\, dV = \frac{1}{2}\dot{q}^T M^i \dot{q}$$

where the integration is over the volume V^i of element i and M^i is the positive definite inertia matrix of element i

$$M^i = \begin{bmatrix} M_{rr}^i & M_{rf}^i \\ (M_{rf}^i)^T & M_{ff}^i \end{bmatrix}$$

where

$$\begin{aligned} M_{rr}^i &= \int_{V^i} \rho_i \begin{bmatrix} I & -A\tilde{r}^i G \\ -(A\tilde{r}^i G)^T & G^T\tilde{r}^{iT}\tilde{r}^i G \end{bmatrix} dV \\ M_{rf}^i &= \int_{V^i} \rho_i \begin{bmatrix} AN^i \\ -G^T\tilde{r}^{iT}N^i \end{bmatrix} dV \\ M_{ff}^i &= \int_{V^i} \rho_i N^{iT}N^i\, dV \end{aligned}$$

and we have used $AA^T = I$. The total kinetic energy of the shaft is $T = \sum_{i=1}^N T^i$. Using virtual work and the relationship between the stress and strain, the potential energy of the ith element due to strain is

$$\begin{aligned} U_s^i &= \frac{1}{2}\int_0^{l_i} EI_i\left[\left(\frac{\partial^2 u_i}{\partial x_i^2}\right)^2 + \left(\frac{\partial^2 v_i}{\partial x_i^2}\right)^2\right] dx_i \\ &= \frac{EI_i}{2} q_f^T B^{iT} \int_0^{l_i} \left[\frac{d^2 S_2^i}{dx_i^2}^T \frac{d^2 S_2^i}{dx_i^2} \right. \\ &\quad \left. + \frac{d^2 S_3^i}{dx_i^2}^T \frac{d^2 S_3^i}{dx_i^2} \right] dx_i B^i q_f \end{aligned}$$

The symmetric constant positive semidefinite stiffness matrix K_{ff}^i of element i is

$$K_{ff}^i = \int_0^{l_i} EI_i B^{iT} \left[\frac{d^2 S_2^i}{dx_i^2}^T \frac{d^2 S_2^i}{dx_i^2} + \frac{d^2 S_3^i}{dx_i^2}^T \frac{d^2 S_3^i}{dx_i^2} \right] B^i\, dx_i$$

The total potential due to strain energy is therefore

$$U_s = \frac{1}{2}q_f^T K_{ff} q_f = \frac{1}{2}q^T \begin{bmatrix} 0 & 0 \\ 0 & K_{ff} \end{bmatrix} q = \frac{1}{2}q^T K q \tag{2}$$

where $K_{ff} = \sum_{i=1}^N K_{ff}^i$. Assuming gravitational force vector is $[0\ \ g\ \ g]^T$ in $OXYZ$, the potential energy due to gravity on the ith element is

$$\begin{aligned} U_g^i &= \int_{V^i} \rho_i \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} r_o^i g\, dV \\ &= \int_{V^i} \rho_i \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \left[R + A \begin{bmatrix} \bar{r}^i \\ u_i + y_i \\ v_i + z_i \end{bmatrix} \right] g\, dV \\ &= \int_{V^i} \rho_i \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \left[R + A \begin{bmatrix} \bar{r}^i \\ N_2^i q_f + y_i \\ N_3^i q_f + z_i \end{bmatrix} \right] g\, dV \end{aligned}$$

The total potential energy due to gravity is

$$U_g = \sum_{i=1}^N U_g^i$$

Using Lagrange's Principle the dynamic model is

$$M\ddot{q} + G\dot{q} + Kq + g_e = F \tag{3}$$

where F is a vector of external forces, $M = \sum_{i=1}^{N} M^i$ is the inertia matrix. $G\dot{q}$ is the vector of centrifugal forces given by

$$G\dot{q} = -\frac{1}{2}\frac{\partial(\dot{q}^T M \dot{q})}{\partial q} + \dot{M}\dot{q} \tag{4}$$

where

$$\frac{\partial(\dot{q}^T M \dot{q})}{\partial q} = \begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1} \\ \dot{q}^T \frac{\partial M}{\partial q_2} \\ \cdot \\ \cdot \\ \cdot \\ \dot{q}^T \frac{\partial M}{\partial q_{4N+6}} \end{bmatrix} \dot{q}$$

$$\dot{M} = \sum_{i=1}^{4N+6} \frac{\partial M}{\partial q_i}\dot{q}_i$$

The force g_e is due to gravity and given by

$$g_e = \begin{bmatrix} g_{re} \\ g_{fe} \end{bmatrix}$$

$$g_{re} = \sum_{i=1}^{N} \begin{bmatrix} 0 \\ m_i g \\ m_i g \\ \int_{V^i} \rho_i \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \frac{\partial A}{\partial \phi} \begin{bmatrix} \bar{r}^i \\ N_2^i q_f + y_i \\ N_3^i q_f + z_i \end{bmatrix} g\, dV \\ \int_{V^i} \rho_i \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \frac{\partial A}{\partial \psi} \begin{bmatrix} \bar{r}^i \\ N_2^i q_f + y_i \\ N_3^i q_f + z_i \end{bmatrix} g\, dV \\ \int_{V^i} \rho_i \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \frac{\partial A}{\partial \theta} \begin{bmatrix} \bar{r}^i \\ N_2^i q_f + y_i \\ N_3^i q_f + z_i \end{bmatrix} g\, dV \end{bmatrix}$$

Defining $\bar{q}_f(i)$ as a $4N \times 1$ vector with its ith element equal to 1 and its other elements 0, then

$$g_{fe} = \sum_{i=1}^{N} \begin{bmatrix} \int_{V^i} \rho_i \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 0 \\ N_2^i \bar{q}_f(1) \\ N_3^i \bar{q}_f(1) \end{bmatrix} g\, dV \\ \int_{V^i} \rho_i \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 0 \\ N_2^i \bar{q}_f(2) \\ N_3^i \bar{q}_f(2) \end{bmatrix} g\, dV \\ \cdot \\ \cdot \\ \cdot \\ \int_{V^i} \rho_i \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 0 \\ N_2^i \bar{q}_f(4N) \\ N_3^i \bar{q}_f(4N) \end{bmatrix} g\, dV \end{bmatrix}$$

In component form (3) is

$$\begin{aligned} &M_{rr}\ddot{q}_r + M_{rf}\ddot{q}_f + Q_{rr}\dot{q}_r \\ &\quad + Q_{rf}\dot{q}_f + g_{re} = F_r \qquad (5) \\ &M_{rf}^T\ddot{q}_r + M_{ff}\ddot{q}_f + Q_{fr}\dot{q}_r + Q_{ff}\dot{q}_f \\ &\quad + K_{ff}q_f + g_{fe} = B_f F_f \qquad (6) \end{aligned}$$

where the external forces due to the magnetic bearings are $F_f = \begin{bmatrix} F_{v,y} & F_{vz} & F_{h,y} & F_{h,z} \end{bmatrix}^T$, $F_r = [F_x \ 0 \ 0 \ D_\omega \ 0 \ 0]^T$ and $B_f = \begin{bmatrix} B_v & B_h \end{bmatrix} = (b_{k,l})_{4N\times 4}$ where B_v, B_f are $4N \times 2$ boolean matrices and D_ω is the electric motor torque. Element $b_{k,l}$ of matrix B_f is related to the position of airgap. We assume the lengths of the elements have been chosen so that the radial airgaps occur at the junction of two elements. If the radial airgaps are located at the junctions of the ith and $(i+1)$th elements and the junction of the jth and $(j+1)$th elements we have

$$b_{k,l} = \begin{cases} 1, & k = 1,2, l = 4(i-1)+k \\ 1, & k = 3,4, l = 4(j-1)+k-2 \\ 0, & \text{otherwise} \end{cases}$$

Next we obtain an expression for the radial airgaps in terms of state q. We use the subscripts v and h to denote the front and rear of the shaft respectively. The X-direction distances of the front and rear airgaps to the origin of $OXYZ$ are denoted $l_{f,v}$ and $l_{f,h}$ respectively. The Y, respectively Z coordinate of the shaft's centerline at the front and rear bearing expressed in $OXYZ$ are denoted $Y_{f,h/v}$ and $Z_{f,h/v}$ respectively. Using (1)

$$\begin{bmatrix} l_{f,h/v} \\ Y_{f,h/v} \\ Z_{f,h/v} \end{bmatrix} = R + Ar = R + A\begin{bmatrix} \zeta \\ B_{h/v}^T q_f \end{bmatrix} \tag{7}$$

$$\begin{aligned} &= \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} \zeta c\psi c\theta \\ \zeta s\theta \\ -\zeta s\psi c\theta \end{bmatrix} + A\begin{bmatrix} 0 \\ B_{h/v}^T q_f \end{bmatrix} \\ &= \begin{bmatrix} X_{p,h/v} \\ Y_{p,h/v} \\ Z_{p,h/v} \end{bmatrix} + \begin{bmatrix} \zeta c\psi c\theta \\ \zeta s\theta \\ -\zeta s\psi c\theta \end{bmatrix} \qquad (8) \end{aligned}$$

where we have defined

$$\begin{aligned} &X_{p,h/v} = X + \\ &\begin{bmatrix} s\psi s\phi - c\psi s\theta c\phi & c\psi s\theta s\phi + s\psi c\phi \end{bmatrix} B_{h/v}^T q_f \\ &Y_{p,h/v} = Y + \begin{bmatrix} c\theta c\phi & -c\theta s\phi \end{bmatrix} B_{h/v}^T q_f \\ &Z_{p,h/v} = Z + \\ &\begin{bmatrix} s\psi s\theta c\phi + c\psi s\phi & -s\psi s\theta s\phi - c\psi c\phi \end{bmatrix} B_{h/v}^T q_f \end{aligned}$$

Solving for ζ in the first equation in (8) and substituting this into the last two equations of (8) gives

$$Y_{f,h/v} = \frac{(l_{f,h/v} - X_{p,h/v})s\theta}{c\psi c\theta} + Y_{p,h/v}, \tag{9}$$

$$Z_{f,h/v} = \frac{(X_{p,h/v} - l_{f,h/v})s\psi}{c\psi} + Z_{p,h/v} \tag{10}$$

which are expressions for the position of the shaft's centre expressed as functions of state q at the radial actuators.

Assuming small ψ, θ we have $\psi \approx s\psi$, $c\psi \approx 1$, $\theta \approx s\theta$, and $c\theta \approx 1$. Thus, the matrices A and G become

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix}$$

and expression (9)–(10) become

$$\begin{aligned} Y_{f,h/v} &= (l_{f,h/v} - X)\theta + Y_{p,h/v} \\ Z_{f,h/v} &= (X - l_{f,h/v})\psi + Z_{p,h/v} \end{aligned}$$

where

$$\begin{aligned} Y_{p,h} &= Y + \begin{bmatrix} c\phi & -s\phi \end{bmatrix} B_h^T q_f \\ Y_{p,v} &= Y + \begin{bmatrix} c\phi & -s\phi \end{bmatrix} B_v^T q_f \\ Z_{p,h} &= Z + \begin{bmatrix} s\phi & c\phi \end{bmatrix} B_h^T q_f \\ Z_{p,v} &= Z + \begin{bmatrix} s\phi & c\phi \end{bmatrix} B_v^T q_f \end{aligned}$$

Defining $\bar{A} = \begin{bmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{bmatrix}$ and $y = \begin{bmatrix} Y_{f,v} & Z_{f,v} & Y_{f,h} & Z_{f,h} \end{bmatrix}^T$, we have

$$y = C_1 q_r + \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{A} \end{bmatrix} B_f^T q_f$$

where

$$C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & l_{fv} - X \\ 0 & 0 & 1 & 0 & X - l_{fv} & 0 \\ 0 & 1 & 0 & 0 & 0 & l_{fh} - X \\ 0 & 0 & 1 & 0 & X - l_{fv} & 0 \end{bmatrix}$$

In order to check a Matlab implementation of the model (5)–(6) we compare the frequencies of a non-rotating free-free uniform cylindrical beam obtained using an analytic solution to the Euler-Bernoulli equation

$$\begin{aligned} m w_{tt}(\xi,t) + EI w_{\xi\xi\xi\xi}(\xi,t) &= 0, \quad \xi \in (0,L) \\ w_{\xi\xi}(0,t) &= 0, \qquad (11) \\ w_{\xi\xi\xi}(0,t) &= 0, \qquad (12) \\ w_{\xi\xi}(L,t) &= 0, \qquad (13) \\ w_{\xi\xi\xi}(L,t) &= 0, \qquad (14) \end{aligned}$$

where EI is rigidity of the beam and w is the displacement field. Using separation of variables $w(\xi,t) = U(\xi)V(t)$ we obtain

$$U^{(4)}(\xi) + \frac{\omega^2 m}{EI} U(\xi) = 0$$

Defining $\lambda = \omega^2 m/(EI)$ and using the boundary conditions (11)–(14) we obtain the following condition on λ:

$$1 - \cos(\lambda L)\cosh(\lambda L) = 0 \tag{15}$$

Numerically solving the roots of (15) gives the first four frequencies $\omega/(2\pi) = 1848, 5093, 9984, 16505$ Hz. Taking the lowest four values of

$$\frac{\sqrt{\text{eig}(M_{ff}^{-1} K_{ff})}}{2\pi}$$

in one direction of motion we obtain $1849, 5109, 10081, 16669$ Hz. Here we used $N = 5$ elements and $\rho = 7800$ kg / m^3, $E = 210$ GPa (steel) and $I = \pi R^4/4$ (circular cross-sectional area) with $R = .2$ m.

CONCLUSION

This paper has presented a nonlinear ordinary differential equation model for a flexible rotor supported by magnetic bearings. The method is readily implemented in Matlab. Future work focusses on validating the model for an actual magnetic bearing test-stand and using the model for control design.

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