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STIFFNESS ANALYSIS OF AXIALLY POLARIZED RADIAL PERMANENT MAGNET BEARINGS

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ABSTRACT
Permanent magnet bearings are important bearings in industrial use. Previous works have only provided a limited analysis of the stiffness of these bearings, usually evaluating the axial and radial stiffness values but neglecting the moment and cross coupled stiffness values. This paper develops a three-dimensional magnetic vector potential solution for the magnetic field. A full 5x5 stiffness matrix is developed for these bearings representing the five degrees of freedom for the three translational and two angular degrees of freedom. A relatively simple set of equations for the stiffness values in terms of the axial bearing stiffness is obtained which is valid over a large range of radial and axial coordinates. Three example calculations for two PM rings forming a radial bearing configuration, three sets of PM rings forming a radial bearing configuration, and a radially and moment stable combination of PM rings are then given. This is a companion paper to "Forces and Moments in Axially Polarized Radial Permanent Magnet Bearings," by Jiang, Allaire, Baloh, and Wood.

INTRODUCTION
Permanent magnet bearings are now commonly employed in many industrial applications. An evaluation of the full three dimensional stiffness values in five axes (not including rotation) requires a 5x5 matrix. There are principal stiffness values in the axial and radial directions plus angular displacement principal stiffness values. There are also many cross coupled stiffness values involving generalized forces due to generalized displacement which have generally not been considered in the analysis of radial PM bearings as reported in the literature to date. This paper is the second part of two separate papers where the first paper is "Forces and Moments in Axially Polarized Radial Permanent Magnet Bearings" [Jiang, et al, 2002].

Yonnet [1981a, 1981b] considered an unrolled bearing configuration to evaluate the stiffness values between two PM bars. He only considered the principal axial and radial stiffness components between the PM rings. Okuda et al [1984] evaluated the axial and radial stiffness of PM rings. Delamare [1994,1995] reports on the stability of a one axis combination of PM rings and an active axial control but does not address the issue of the complete stiffness matrix for PM rings. Marinescu and Marinescu [1994] evaluated the axial and radial stiffness of PM ring bearings using a two dimensional magnetic vector potential. Ohji et al [1999] presented an experimental study of the radial disturbance attenuation of a rotor in PM bearings and some eddy current loss measurements. All of these works evaluated both radial and axial stiffnesses of permanent magnet bearings but did not evaluate the full stiffness matrix for the PM bearings.

This paper develops a general stiffness matrix for a ring magnet or set of ring magnets which includes all five degrees of freedom - principal stiffness and cross coupled stiffness values. The evaluation of the radial and axial stiffness terms in this work includes terms up to second order in radial displacement to yield a very accurate solution. This leads to a general 5x5 stiffness matrix which must be evaluated of the form

$$K = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} & K_{x\alpha} & K_{x\beta} \\ K_{yx} & K_{yy} & K_{yz} & K_{y\alpha} & K_{y\beta} \\ K_{zx} & K_{zy} & K_{zz} & K_{z\alpha} & K_{z\beta} \\ K_{\alpha x} & K_{\alpha y} & K_{\alpha z} & K_{\alpha\alpha} & K_{\alpha\beta} \\ K_{\beta x} & K_{\beta y} & K_{\beta z} & K_{\beta\alpha} & K_{\beta\beta} \end{bmatrix} \quad (1)$$

There are 25 stiffness values to be determined. However, this matrix is symmetric so only 15 stiffness values are unique. All of these must be evaluated. It is also desired evaluate the stiffness values as a function of the coordinate position (r,z) at any position with the range of the airgap thickness rather than simply obtaining a single stiffness value for the centered position.

NOMENCLATURE

Same as "Forces and Moments in Axially Polarized Radial Permanent Magnet Bearings" [Jiang, et al, 2002].

AXIAL STIFFNESS

Fig. 1 shows the geometry with two PM rings forming a magnetic bearing with the geometry of the two axisymmetric PM rings, one of radius a, height h_a, current density J_a and the other of radius b, height h_b, and current density J_b. The axial force between the rings is given by Jiang et al [2002]. The axial stiffness is given by

$$K_{zz}(z) = -\frac{\partial F(z)}{\partial z} = \frac{\mu_0 J_a J_b}{2}\left[\frac{\partial f(z_b+z+h_b)}{\partial z} - \frac{\partial f(z_b+z)}{\partial z}\right]$$
$$= \frac{\mu_0 J_a J_b}{2}[k(z_b+z+h_b-z_a-h_a) - k(z_b+z+h_b-z_a)$$
$$-k(z_b+z-z_a-h_a) + k(z_b+z-z_a)] \quad (2)$$

where $k(z)$ is

$$k(z) = ba\int_0^{2\pi}\frac{\cos(\phi_a)}{\left(a^2+b^2-2ab\cos(\phi_a)+z'^2\right)^{\frac{1}{2}}}d\phi_a \quad (3)$$

The integrals in (3) are easily evaluated using a mathematical code such as *Mathematica.*

If there is a series of the PM rings, the axial stiffness is obtained by the sum or superposition of all of the stiffness values. The solution is easily found as a sum of elliptic integrals also using *Mathematica* and numerical integration.

PRINCIPAL RADIAL STIFFNESS

The radial stiffness is determined by taking a small displacement of one magnet ring in the radial direction of a distance r, as shown in Fig. 2..

The stiffness term is

$$K_{rr}(\bar{r}) = -\frac{1}{b}\frac{\partial F_r(\bar{r},\bar{z})}{\partial \bar{r}} = o(\bar{r}^2) - \frac{1}{b}$$
$$\int_{\bar{z}_b+\bar{z}}^{\bar{z}_b+\bar{z}+\bar{h}_b} J_b b\,\pi[A_a(1,\bar{z}^T) - \frac{\partial A_a(1,\bar{z}^T)}{\partial \bar{b}'} - \frac{\partial^2 A_a(1,\bar{z}^T)}{(\partial \bar{b}')^2}]d\bar{z}^T \quad (4)$$

where the radial force was obtained in Jiang et al [2002]. If this equation is evaluated at the centered position r=0,z=0, the principal radial stiffness is constant. However, this expression holds for non centered values of radial and axial position. The integrals are easily evaluated using a mathematical code such as *Mathematica* but this is not necessary as the values can be related to the axial stiffness. If r=0, then the radial stiffness is exactly half of the axial stiffness. And the above equation the constant part of radial stiffness is equal to half of axial stiffness.

Thus, from Equation (4) radial stiffness equations become

$$K_{rr}(\bar{r},\bar{z}) = -\frac{K_{zz}(\bar{z})}{2} + o(\bar{r}^2) \quad (5)$$

The force is aligned with $\bar{r}$ and proportional to $\bar{r}$. Thus, the principal stiffness components K_{xx},K_{yy} are given by the following formula.

$$K_{xx} = K_{yy} = -\frac{1}{2}K_{zz}(\bar{z}) \quad (6)$$

They are exactly one half of the axial stiffness but of different sign.

CROSS COUPLED RADIAL STIFFNESS

The cross coupled stiffnesses values are easily found to be zero.

$$K_{xy} = K_{yx} = 0 \quad (7)$$

This is because the co-energy is only related to the displacement of the ring in x or y coordinates so there is no cross product.

CROSS COUPLED AXIAL-RADIAL STIFFNESS

The cross coupled axial-radial stiffness are given by

$$K_{rz} = K_{zr} = -\frac{\partial F_r(\bar{r},z)}{\partial z} = -\frac{r}{2}\frac{\partial K_{zz}}{\partial z} + o(\bar{r}^3) \quad (8)$$

where their stiffness values are proportional to the radial displacement. When the rotor is centered, these xz, yz plane cross-coupled stiffness terms are zero and have the largest value when the rotor is at the largest radial position in the bearing clearance space. Generally these components are small compared to the principal stiffness terms.

MOMENT STIFFNESS

The moment stiffness acting on ring b caused by the attraction between a and b due to an angular displacement can also be determined. As shown in Fig. 3, the diagonal angular stiffness is given by

$$K_{\alpha\alpha} = K_{\beta\beta} = -\frac{\partial M_a}{\partial \alpha} = o(\alpha^2) -$$
$$b\int_{\bar{z}_b+\bar{z}}^{\bar{z}_b+\bar{z}+\bar{h}_b} J_b b\,\pi\left[A_a + \frac{\partial A_a}{\partial \bar{b}'} - \frac{\partial^2 A_a}{(\partial \bar{z}')^2}\right]_{\alpha=0} d\bar{z}^T \quad (9)$$

where the moment equation is given in Jiang et al [2002]. It can be shown that the most important third term in this expansion is exactly $b^2 K_{zz}/2$ where b is the ring diameter. Also, it can be shown that the first two terms only reduce the stiffness by approximately 4% to 5% in most case. Thus, the two angular stiffnesses can be approximately given by

$$K_{\alpha\alpha} = K_{\beta\beta} \approx \frac{b^2}{2} K_{zz} \quad (10)$$

This means that if axial stiffness is positive, than the diagonal angular stiffness values are positive. This expansion is accurate to relatively high order in alpha as shown in (9). This derivation considered the rings to be radially centered so a small effect when there is some radial displacement was not considered.

CROSS COUPLED ANGULAR STIFFNESS

It can also be shown from the above formulas that the cross coupled angular displacement stiffnesses vanishes. Thus

$$K_{\alpha\beta} = K_{\beta\alpha} = 0 \quad (11)$$

to be employed in the stiffness matrix.

CROSS COUPLED AXIAL-ANGULAR STIFFNESS

The axial-angular stiffness can easily shown to be

$$K_{\alpha z} = K_{z\alpha} = -\frac{b^2}{2} \frac{\partial K_{zz}}{\partial z} \alpha \quad (12)$$

where Hc is the distance between moving ring b center and moment reference center. (Fig 3)

CROSS COUPLED RADIAL-ANGULAR STIFFNESS

The radial-angular cross coupled stiffness terms are

$$K_{x\alpha} = K_{\alpha x} = K_{y\beta} = K_{\beta y} = 0 \quad (13)$$

from the above magnetic vector potential equations. The remaining cross coupled geometry stiffness terms are

$$K_{\alpha y} = K_{y\alpha} = -H_C K_{yy} = \frac{H_C}{2} K_{zz} \quad (14)$$

where Hc is given above. (Fig 2)

STIFFNESS MATRIX

The PM ring stiffness matrix from Eq. (1) is given by

$$\begin{bmatrix} -\frac{K_{zz}}{2} & 0 & -\frac{x}{2}\frac{\partial K_{zz}}{\partial z} & 0 & \frac{H_C K_{zz}}{2} \\ 0 & -\frac{K_{zz}}{2} & -\frac{y}{2}\frac{\partial K_{zz}}{\partial z} & \frac{H_C K_{zz}}{2} & 0 \\ -\frac{x}{2}\frac{\partial K_{zz}}{\partial z} & -\frac{y}{2}\frac{\partial K_{zz}}{\partial z} & K_{zz} & -\frac{b^2}{2}\frac{\partial K_{zz}}{\partial z}\alpha & -\frac{b^2}{2}\frac{\partial K_{zz}}{\partial z}\beta \\ 0 & \frac{H_C K_{zz}}{2} & -\frac{b^2}{2}\frac{\partial K_{zz}}{\partial z}\alpha & \frac{b^2 K_{zz}}{2} & 0 \\ \frac{H_C K_{zz}}{2} & 0 & -\frac{b^2}{2}\frac{\partial K_{zz}}{\partial z}\beta & 0 & \frac{b^2 K_{zz}}{2} \end{bmatrix} \quad (15)$$

Here all stiffness components are related to the axial stiffness term by simple constants, geometric parameters, or the derivative of the axial stiffness. Some complicated terms are not shown completely due to space considerations. It should be noted that these stiffness values are all functions of the radial, axial, angular position - not simply for the centered case.

RADIAL BEARING COMPOSED OF TWO PM RINGS (EXAMPLE 1)

A set of two magnetic rings is given in Marinescu and Marinescu [1994]. As shown in Fig. 4, these have an inner radius of 10 mm, a radial air gap of 4 mm, height of 10 mm and a radial thickness of 10mm, where the inner rings are the moving components and the outer rings are fixed. The ring diameters are a=68mm and b=20mm and the ring heights are h_a=h_b=10mm. The rings are axially magnetized with a flux density of 1.1 T. The parameter Hc=0. The stiffness components are calculated by the method developed in this paper and results given in Table 1. The calculated axial stiffness is Kzz=-27.8N/mm and the x and y stiffness values are given as Kxx=Kyy=13.88N/mm, in excellent agreement with Marinescu and Marinescu [1994] who gave the value of radial stiffness Kxx=Kyy=13.95N/mm. The full 5x5 stiffness matrix is given in Table 1 where the variation of stiffness with all coordinates x, y, z, α, β is included but operating about the position z=2mm. If z=0mm and the rotor is perfectly centered, all of the off diagonal stiffness terms vanish. This analysis represents a significant extension of Marinescu [1994] in evaluating both moment effects and the variation of the stiffness with rotor position in the clearance space.

RADIAL BEARING WITH THREE PM RINGS (EXAMPLE 2)

A more complicated set of three axially polarized magnetic rings is shown in Fig. 5. These are composed of rings of 4 mm x 4 mm cross section, with each coaxial half oppositely polarized, and spacing of 0.4 mm between rings. The radius of the inner ring is 13 mm. The rings are axially magnetized with a flux density of 1.1 T. The example stiffness values are given in Table 2 at an axial operating position of z=0.2mm. It is seen that the stiffness values are much higher than in Example 2 due to the larger size rings and the use of three stacked rings. The cross coupled stiffness values are shown as well. If the rings are axially centered where z=0, all cross coupling terms are zero and the stiffness matrix is diagonal.

STABLE RADIAL AND MOMENT COMBINED BEARING (EXAMPLES 3 & 4)

Consider two sets of rings employed in Delamare et al [1994] which are stable in radial stiffness and moment, thus having only one axis which needs to be actively controlled. Fig. 6 shows the geometry with the axial magnetic polarization indicated by the arrows. The four inner magnet rings are 2 mm x 2 mm each, the inner radius of the two rings is 2 mm, and the axial airgap thickness is 0.5 mm. These four rings provide positive radial stiffness. The two outer rings have dimensions 1 mm x 1 mm, the outer radius of the inner ring is 12 mm, and the radial gap between the two rings is 0.7 mm. The PM material has magnetic flux density of B=1.25Tesla. The radial and axial stiffness values are given in Table 3 for the centered case. The cross coupled stiffness terms which involve the z coordinate are linear with the displacement or angular displacement as indicated in Table 3. Table 4 gives the results when the axial position is moved upward to the location z=0.2mm. It is easily seen that many of the stiffness values are constant but the terms involving z change significantly.

This PM magnet configuration has positive radial and moment stiffness but large negative axial stiffness. Thus, it can be used for a PM suspension where only the axial axis need be actively controlled. It is also seen that the stiffness matrix has relatively large cross coupled stiffness values which relate the x translation rotor motions to the alpha tilt rotor motions and the y translation rotor motions to the beta tilt rotor motions.

CONCLUSIONS

An analytical expression has been developed for permanent ring magnet configurations using the magnetic vector potential formulation. The full 25x25 symmetric stiffness matrix has been obtained. Previous published results such as those due to Yonnet [1981a,b] and Marinescu [1994] only evaluated axial and radial stiffness but did not evaluate moment stiffness. It was shown that some of the cross coupled stiffness values are identically zero and some are nearly constant. Others vary with the radial or angular position of one ring relative to the other. The analysis is suitable for evaluation by commonly available mathematical programs such as Matlab. Three example ring configurations were evaluated and results presented.

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FIGURES

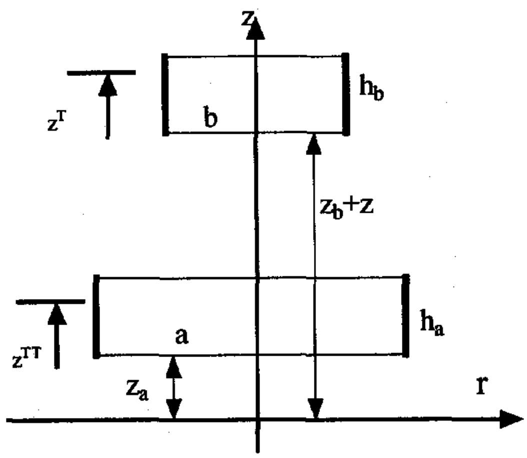

FIG. 1: Axial Stiffness Geometry

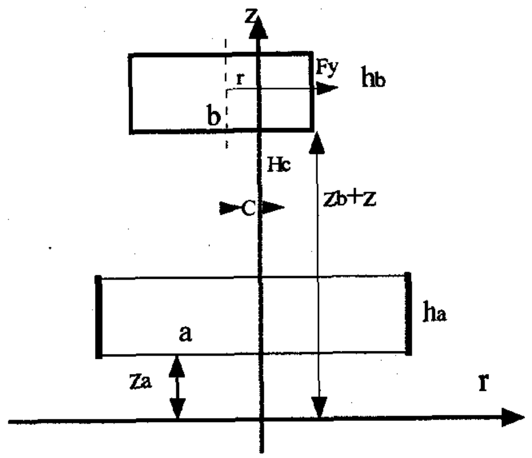

FIG. 2: Radial Stiffness Geometry

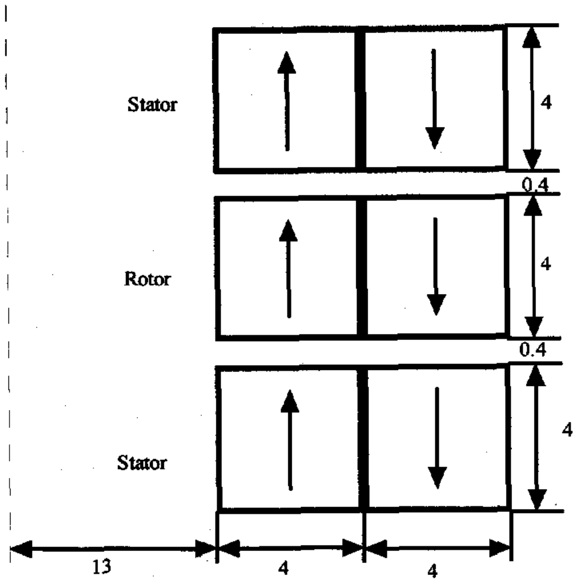

FIG. 5: Example 2, stack bearing

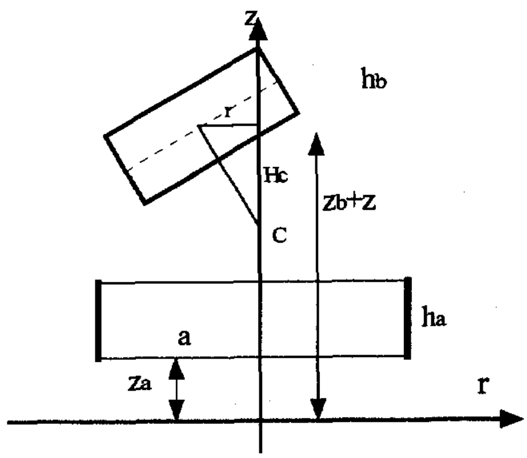

FIG. 3: Moment Stiffness Geometry

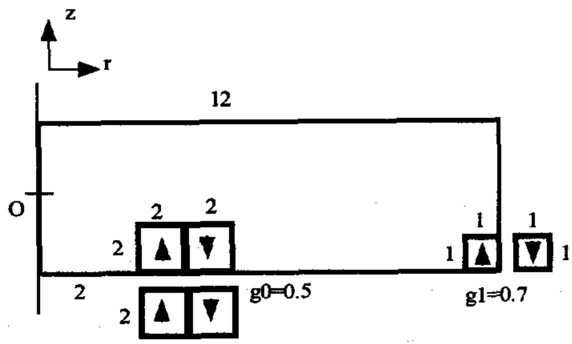

FIG. 6: Example 3, Radial and Moment stable bearing

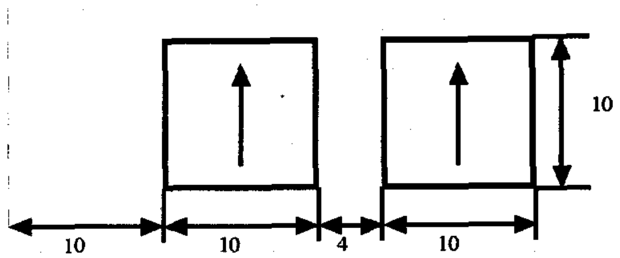

FIG. 4: Example 1, two ring bearing

TABLE 1: Result of Example 1, z=2mm

	X	Y	Z	α	β
X	13.87 N/mm	0	-2.06x N/mm	0	0
Y	0	13.87 N/mm	-2.06y N/mm	0	0
Z	-2.06x N/mm	-2.06y N/mm	-27.75 N/mm	846α Nm/rad	846β Nm/rad
α	0	0	846α Nm/rad	-8.357 Nm/rad	0
β	0	0	846β Nm/rad	0	-8.357 Nm/rad

TABLE 2: Result of Example 2, z=0.2mm

	X	Y	Z	α	β
X	175 N/mm	0	163x N/mm	0	0
Y	0	175 N/mm	163y N/mm	0	0
Z	163x N/mm	163y N/mm	-351 N/mm	-5E4α Nm/rad	-5E4β Nm/rad
α	0	0	-5E4α Nm/rad	-53 Nm/rad	0
β	0	0	-5E4β Nm/rad	0	-53 Nm/rad

TABLE3: Result of Example 3, z=0mm

	X	Y	Z	α	β
X	9.5 N/mm	0	-24.4x N/mm	0	6.9(N)
Y	0	9.5 N/mm	-24.4y N/mm	6.9(N)	0
Z	-24.4x N/mm	-24.4y N/mm	-19 N/mm	511α Nm/rad	511β Nm/rad
α	0	6.9(N)	511α Nm/rad	0.513 Nm/rad	0
β	6.9(N)	0	511β Nm/rad	0	0.513 Nm/rad

TABLE4: Result of Example 4, z=0.2mm

	X	Y	Z	α	β
X	6.1 N/mm	0	-10.5x N/mm	0	3.8(N)
Y	0	6.1 N/mm	-10.5y N/mm	3.8(N)	0
Z	-10.5x N/mm	-10.5y N/mm	-12.2 N/mm	-452α Nm/rad	-452β Nm/rad
α	0	3.8(N)	-452α Nm/rad	0.32 Nm/rad	0
β	3.8(N)	0	-452β Nm/rad	0	0.32 Nm/rad