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FAST CALCULATION METHOD FOR THE FORCES AND STIFFNESSES OF PERMANENT-MAGNET BEARINGS

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ABSTRACT

An analytical method for the calculation of the forces and the stiffnesses of permanent-magnet bearings using complete elliptic integrals is presented. Due to the rotational symmetry a permanent-magnet bearing can be described by a cylindrical coordinate system. With this assumption the axial force and four stiffness numbers have to be regarded only. The magnetic field of an axially magnetized ring (hollow cylinder) can be described using complete elliptic integrals of first, second and third kind. For this type of integrals fast and accurate algorithms are known. A high-speed computer code using these algorithms has been developed to calculate the relevant force and stiffness values. This computer code is the basis for the optimization of permanent-magnet bearings.

INTRODUCTION

For the design of permanent magnet bearings the forces and the stiffnesses are important. However, using commercial software it is difficult to obtain the stiffness in a fast and accurate way, since it usually only provides force and torque calculation. Stiffnesses are derived only in an indirect way. The force difference due to small displacements leads to stiffness. For that reason the force calculations has to be done twice. This method is time consuming with low accuracy. That is why for permanent-magnet bearings a fast and accurate model is presented using the following assumptions:

- rotational symmetry
- relative permeability $\mu_r = 1$
- no soft magnetizable material e.g. iron
- axially magnetized rings

The limitation on the rotational symmetry is not a real limit for bearings. But as a consequence only one force and four stiffness numbers have to be considered. The second assumption (μ_r=1) holds for the rare-earth-materials used for industrial permanent magnets with high magnetization. The third assumption is more restrictive, because bearings using the force between permanent magnets and iron (reluctance bearings) are excluded. In that case the calculation can be done only with numerical methods (FEM, BEM).

BASIC IDEAS

The magnetic field of a homogen magnetized permanent magnet can be calculated assuming surface currents $\mathbf{K}$ at the lateral surface of the magnet. It can be obtained from the

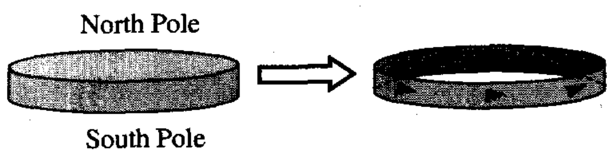

Figure 1: Model of a permanent magnet with surface currents

remanence $\mathbf{B_r}$

$$\mathbf{K} = \frac{1}{\mu_0}\mathbf{B_r}. \tag{1}$$

However, this simple rule is valid for $\mu_r = 1$. The vector potential $\mathbf{A}$ can be derived as follows

$$\mathbf{A}(\mathbf{x_2}) = \frac{\mu_0}{4\pi} \int_A \frac{\mathbf{K}(\mathbf{x_1})}{|\mathbf{r}|} dA \qquad (2)$$

with $\mathbf{r} = \mathbf{x_2} - \mathbf{x_1}$. The derivation (curl) of the vector potential results in the magnetic flux density

$$\mathbf{B} = \text{rot}\mathbf{A}\,. \qquad (3)$$

Both steps (2) and (3) can be summarized into one equation known as the Biot-Savart law

$$\mathbf{B}(\mathbf{x_2}) = \frac{\mu_0}{4\pi} \oint \frac{\mathbf{K}(\mathbf{x_1}) \times \mathbf{r}}{|\mathbf{r}|^3} dA\,. \qquad (4)$$

Using this field equation, the force is calcu-

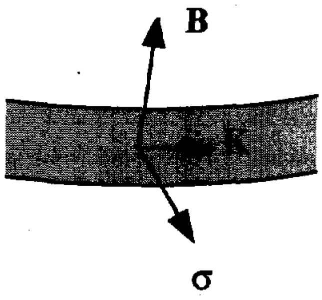

Figure 2: Lorentz-law

lated by the Lorentz-law

$$\boldsymbol{\sigma} = \mathbf{K} \times \mathbf{B}\,. \qquad (5)$$

Together with the *external* magnetic field, surface currents generate mechanical stresses $\boldsymbol{\sigma}$ on the surface area. The forces are obtained by surface integration of the stress

$$\mathbf{F} = \int (\mathbf{K} \times \mathbf{B}) dA\,. \qquad (6)$$

Finally, we get the stiffnesses simply by derivation of the forces, that is

$$\mathbf{S_{PM}} = -\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\,. \qquad (7)$$

Here we only consider axial magnetized hollow cylinders. Due to the rotational symmetry, the above equations reduce significantly. For example, the vector of the surface currents consists only of one component in circumferential direction K_φ. The same follows for the vector potential A_φ. Since the flux density $\mathbf{B}$ is the curl of the vector potential it consists of two components

$$B_\rho = -\frac{dA_\varphi}{dz}\,, \qquad (8)$$

$$B_z = \frac{1}{\rho}\frac{d\rho A_\varphi}{d\rho}\,. \qquad (9)$$

THE AXIAL FORCE

Due to the rotational symmetry only axial force F_z occurs. So we must only consider the axial stress component in equation xx. The cross product in cylindrical coordinates yields

$$\sigma_z = -K_\varphi B_\rho\,. \qquad (10)$$

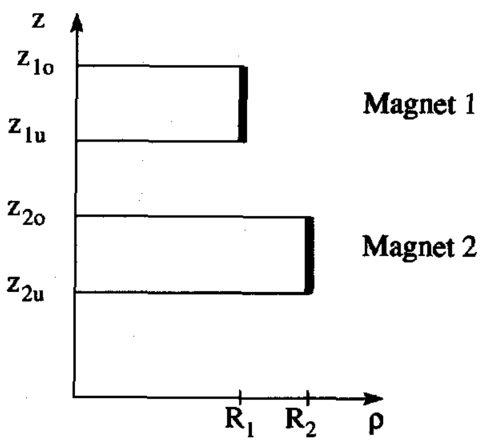

Figure 3: Cross section of two magnets

To obtain the axial force on magnet 2 this expression must be integrated both over the circumference and the height of the lateral surface

$$F_z = \int_{z_{2u}}^{z_{2o}} \int_0^{2\pi} -K_\varphi B_\rho R_2 d\varphi_z dz\,. \qquad (11)$$

After the simple integration over the circumference we get

$$F_z = -K_\varphi 2\pi R_2 \int_{z_{2u}}^{z_{2o}} B_\rho(R_2, z) dz\,. \qquad (12)$$

The radial component of the flux density can be expressed as the derivation of the vector potential (equation 9)

$$F_z = 2\pi R_2 \cdot K_\varphi \int_{z_{2u}}^{z_{2o}} \frac{dA_\varphi}{dz} dz \qquad (13)$$

$$= 2\pi R_2 \cdot K_\varphi A_\varphi(R_2, z)|_{z_{2u}}^{z_{2o}} . \qquad (14)$$

As final expression for the **axial force** on magnet 2, we obtain

$$F_z = 2\pi R_2 K_\varphi \left[A_\varphi(R_2, z_{2o}) - A_\varphi(R_2, z_{2u})\right] . (15)$$

The axial force is essentially a function of the vector potential while the surface current is a constant value. Hence, an analytical expression for the vector potential A_φ is needed.

THE STIFFNESS MATRIX

In general, the derivation of the force vector leads to a complicated stiffness matrix (eq. 7). But due to the rotational symmetric model and the conservative nature of the magnetic field the matrix gets the following form [1]

$$\mathbf{S}_{PM} = \begin{pmatrix} s_r & 0 & 0 & 0 & s_{\varphi r} & 0 \\ 0 & s_r & 0 & -s_{\varphi r} & 0 & 0 \\ 0 & 0 & s_{ax} & 0 & 0 & 0 \\ 0 & -s_{\varphi r} & 0 & s_{\varphi\varphi} & 0 & 0 \\ s_{\varphi r} & 0 & 0 & 0 & s_{\varphi\varphi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \qquad (16)$$

This matrix contains four stiffness numbers only. Further applying the Earnshaw Theorem we can express the radial stiffness s_{rad} in terms of the axial stiffness. The Earnshaw Theorem sounds for the rotational symmetric case as

$$s_{rad} = -\frac{1}{2} s_{ax} . \qquad (17)$$

This shows that the stiffness matrix consists of only three independent stiffness numbers, namely

$$\text{axial stiffness:} \quad s_{ax} = -\frac{\Delta F_z}{\Delta z} , \qquad (18)$$

$$\text{tilt stiffness:} \quad s_{\varphi\varphi} = -\frac{\Delta M_x}{\Delta \varphi_x} , \qquad (19)$$

$$\text{coupling stiffness:} \quad s_{\varphi r} = -\frac{\Delta F_x}{\Delta \varphi_y} . \qquad (20)$$

We derive the axial stiffness from the axial force as follows

$$s_{ax} = -\frac{dF_z}{dz} \qquad (21)$$

$$= 2\pi R_2 K_\varphi \frac{d\left[A_\varphi(R_2, z_{2o}) - A_\varphi(R_2, z_{2u})\right]}{dz} .$$

Applying eqn. 8 we get the **Axial Stiffness** on magnet 2

$$s_{ax} = 2\pi R_2 K_\varphi \left[B_\rho\left(R_2, z_{2o}\right) - B_\rho\left(R_2, z_{2u}\right)\right] . \qquad (22)$$

To calculate the axial stiffness an analytical expression for the radial component of the flux density is required.

For the tilt and the coupling stiffnesses similar equations can be derived. But this is a more extensive procedure, so that we consider the axial stiffness only.

THE MAGNETIC FIELD

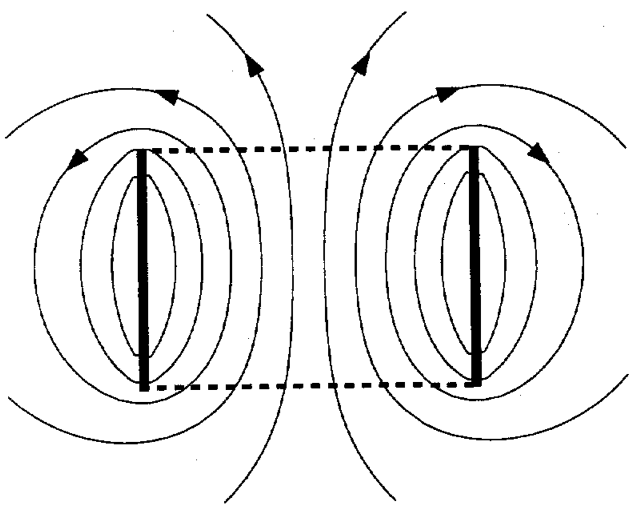

Figure 4: The magnetic field of a cylinder

As shown above in a rotational symmetric model the vector potential and the magnetic flux density are required. Analytical equations are available for the magnetic field of a hollow cylinder as shown by T. Kolbenheyer in 1964 [2], and later adopted by Beyer [3] and Urankar [4]. Here we present the equations for the magnetic flux density and the vector potential in our own notation which is slightly different from the above mentioned.

Vector potential

$$A_\varphi(\rho,z) = -\frac{\mu_0 K_1}{4\pi}\left[\frac{z_{1o}}{\rho}\left(\frac{z_{1o}^2+2R_1^2+2\rho^2}{\sqrt{c_o}}K(k_o)-\sqrt{c_o}E(k_o)-\frac{(R_1-\rho)^2}{c_o}\Pi(\lambda,k_o)\right)\right.$$
$$\left.-\frac{z_{1u}}{\rho}\left(\frac{z_{1u}^2+2R_1^2+2\rho^2}{\sqrt{c_u}}K(k_u)-\sqrt{c_u}E(k_u)-\frac{(R_1-\rho)^2}{c_u}\Pi(\lambda,k_u)\right)\right]; \quad (23)$$

Magnetic Flux Density
Radial component

$$B_\rho = \frac{\mu_0 K_1}{4\pi}\cdot\left[\frac{4R_1}{\sqrt{c_o}}\left\{2\frac{K(k_o)-E(k_o)}{k_o^2}-K(k_o)\right\}-\frac{4R_1}{\sqrt{c_u}}\left\{2\frac{K(k_u)-E(k_u)}{k_u^2}-K(k_u)\right\}\right]; (24)$$

Axial component

$$B_z = -\frac{\mu_0 K_1}{4\pi}\left[\frac{2z_{1o}}{\sqrt{c_o}}\left\{K(k_o)+\frac{R_1-\rho}{(R_1+\rho)}\Pi(k_o^2,\lambda_o^2)\right\}-\frac{2z_{1u}}{\sqrt{c_u}}\left\{K(k_u)+\frac{R_1-\rho}{(R_1+\rho)}\Pi(k_u^2,\lambda_u^2)\right\}\right]; \quad (25)$$

with

$$c_o = (R_1+\rho)^2+z_{1o}^2$$
$$c_u = (R_1+\rho)^2+z_{1u}^2. \quad (26)$$

Modulus k is defined as

$$k_o^2 = \frac{4R_1\rho}{(R_1+\rho)^2+z_{1o}^2} \quad (27)$$

$$k_u^2 = \frac{4R_1\rho}{(R_1+\rho)^2+z_{1u}^2} \quad (28)$$

The parameter λ reads as

$$\lambda^2 = \frac{4R_1\rho}{(R_1+\rho)^2}. \quad (29)$$

$E(k), K(k), \Lambda(k,\lambda)$ are the complete elliptic integrals, as described below.

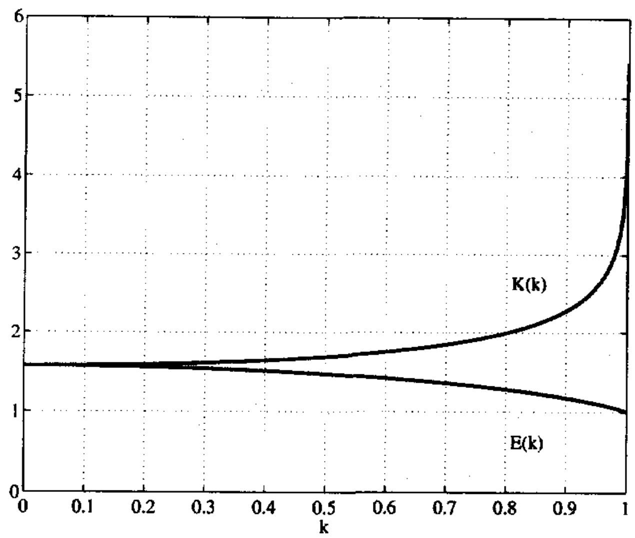

Figure 5: Complete Elliptic Integrals of First Kind K(k) and Second Kind E(k)

COMPLETE ELLIPTIC INTEGRALS

In the equation for the vector potential and for the flux density arise expressions called Elliptic Integrals. There are three kinds of complete elliptic integrals.

1. *Complete elliptic integral of first kind*

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}} \quad (30)$$

2. *Complete elliptic integral of second kind*

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2 sin^2\theta}\, d\theta \quad (31)$$

The integrals are shown in Figure 5.

3. *Complete elliptic integral of third kind*

$$\Pi(k,\lambda) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-\lambda^2 sin\theta)\sqrt{1-k^2 sin^2\theta}} \quad (32)$$

For complete elliptic integrals fast and accurate algorithms are availabele. The algorithms for the calculation of $K(k)$ and $E(k)$ can be found in [5]. For $\Pi(k,\lambda)$ an efficient code can be found in [6, 7] . The used algorithms are given in the Appendix in the MATLAB-Language. The results of these algorithms were compared with the tables in [8]. We notice that power series, which are

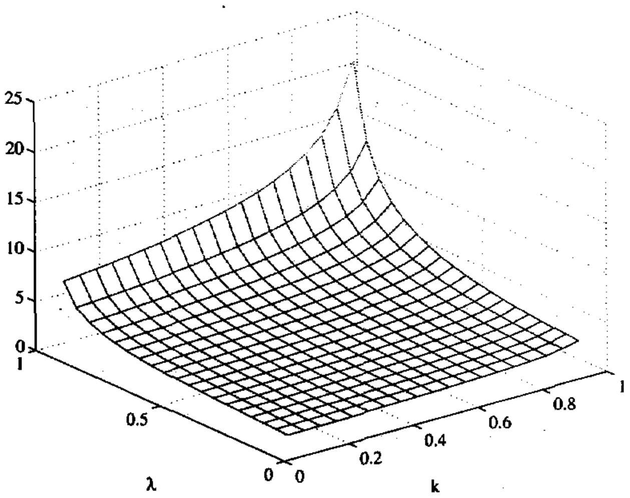

Figure 6: Complete Elliptic Integral of Third Kind $\Pi(k, \lambda)$

found in some mathematical handbooks [8, 9] are often not suitable for numerical calculation, because they show poor convergence especially for $k \approx 1$.
The following table shows the comparison between the described analytical method and the commercial numerical program FARADAY. The force and stiffnesses between two ring magnets with the same size are caclulated. The magnet dimensions are:
Outer radius: 2m, Height: 1m
Distance between the magnets: 1m
Surface current: 1A/m.

Table 1: Force and Stiffnesses between two ring magnets, calculated with the analytical method and the numerical method

	analytical	numerical
F_z	$7.8546 \cdot 10^{-7}$N	$7.8546 \cdot 10^{-7}$N
s_{ax}	$-7.5705 \cdot 10^{-7}$	$-7.5443 \cdot 10^{-7}$
$s_{\varphi\varphi}$	$1.54 \cdot 10^{-8}$	$1.5517 \cdot 10^{-8}$
$s_{\varphi r}$	$-8.1543 \cdot 10^{-7}$	$-8.1548 \cdot 10^{-7}$

CONCLUSIONS

An analytical method using elliptical integrals to calculate the axial force and the axial stiffness is presented. The algorithms are fast and give accurate results. This computer code was used as a basis for the optimization of permanent-magnet bearings.

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APPENDIX

Complete Elliptic Integral 1.Kind

```
function y=compelli1(x)
%Complete Elliptic Integral 1.Kind: K(k)
%Input Range: 0<= x<=1
y=1;

if(x<1)
 ari=1;
 geo=sqrt(1-x^2);
 while(abs(ari-geo)>1e-15*ari)
       ari_neu=(ari+geo)/2;
       geo_neu=sqrt(ari*geo);
       ari=ari_neu;
       geo=geo_neu;
 end %while
 y=pi/2/ari;
else
 y=inf;
end %if
```

Complete Elliptic Integral 2.Kind

```
function y=compelli2(x)
%Complete Elliptic Integral 2.Kind: E(k)
%Input Range: 0<=x<=1
y=0.5*pi;
Apara=1;
Bpara=1-x^2;
ari=1;
geo=sqrt(Bpara);
if (x<1)
 diff=1;
 while (diff>1e-14)
       diff=abs(1-geo/ari);
       ari_neu=(ari+geo);
       geo_neu=2*sqrt(ari*geo);
       Apara_neu=Apara+Bpara/ari;
       Bpara_neu=2*(Bpara+geo*Apara);
       ari=ari_neu;
       geo=geo_neu;
       Apara=Apara_neu;
       Bpara=Bpara_neu;
 end
 Apara_neu=Apara+Bpara/ari;
 y=pi*Apara_neu/(4*ari);
end%if
```

Complete Elliptic Integral 3.Kind

```
function y=compelli3(k2,n)
% Complete Elliptic Integral 3.Kind P(k^2,n)
% Input Range: 0<=k2<=1
%              0<=n<=1 n=lambda^2
% special cases: P(k^2,0)=K(k)
%                P(k^2,1)=inf
%                P(1,n)=inf
ca=10e-16; %accuracy
p=1-n;
kc=sqrt(1-k2);
if(p*kc~=0)     %n=1 or k=1
    m0=1;
    e0=kc;
    if(p>0)     %n<1
       p0=sqrt(p);
       c0=1;
       d0=1/p0;
    else      %n>1
       g=1-p;
       f=kc*kc;
       p0=sqrt(f/g);
       c0=0;
       d0=-k2/(g*p0);
    end;
    while (abs(1-e0/m0)>ca)
       m1=e0+m0;
       e1=2*sqrt(e0*m0);
       p1=e0*m0/p0+p0;
       c1=d0/p0+c0;
       d1=2*(e0*m0/p0*c0+d0);

       m0=m1;
       e0=e1;
       p0=p1;
       c0=c1;
       d0=d1;
    end;
    y=0.5*pi*(c0*m0+d0)/(m0*(m0+p0));
else
    y=inf;
end;
```