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# TUNEABLE $H_\infty$ VIBRATION CONTROL IN ROTOR/MAGNETIC BEARING SYSTEMS USING PARAMETER DEPENDENT SHAPING FUNCTIONS

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## ABSTRACT

In this paper, a method is presented for the derivation of tuneable multivariable controllers for application to a magnetic bearing/rotor system. Controller design specifications are considered that use $H_\infty$ optimisation criteria with linear parameter dependent shaping functions. Linear matrix inequality (LMI) techniques are used to synthesise controllers that satisfy the $H_\infty$ design specification over the range of parameter values using a convex interpolation of vertex controllers. In this way the design parameters that determine the shaping functions can be adjusted on-line to tune the performance and robustness characteristics of the controller. Specific performance measures that are considered are those of rotor synchronous vibration levels, transmitted force levels, base motion rejection, as well as controller stability margins. The effectiveness of such an approach for the synthesis of tuneable $H_\infty$ controllers for a rotor/magnetic bearing system is evaluated.

## 1. INTRODUCTION

Tuneable controllers compare favourably with fixed controllers that are selected on the basis of their predicted performance, which may be poor in practice due to modelling inaccuracies. In the case of rotor/magnetic bearing systems, such inaccuracies may arise in a theoretical model of the system due to discretisation and lumped parameter approximations. It is therefore advantageous to be able to fine-tune a controller on-line to improve performance in the presence of such uncertainties. A simple example would be the tuning of proportional and derivative feedback in PID control to give an acceptable combination of bearing stiffness and modal damping in the closed loop system. In addition, robustness problems, such as instability of unmodelled rotor flexural modes can be avoided if controller gains can be tuned on-line. Incorporating tuneable parameter dependence is straightforward for low order controllers, which are only dependent on a small number of parameters. However, the formulation of high order parameter dependent optimal controllers presents a more difficult problem that will be the focus of the paper.

As well as allowing the fine-tuning of performance, a parameter dependent controller allows the control law to be varied according to the state of operation. For example, the dynamics of the plant may be dependent on a number of time varying parameters. Therefore an appropriate control law that is also dependent on these parameters will enable good performance to be achieved over the entire range of operating conditions. When the plant parameters are measured or estimated in real time and used to automatically adjust the control law this is termed gain scheduling. An obvious application of gain scheduling in rotor/magnetic bearing systems is the use of a controller that is a function of the rotational speed. Mason *et al.* [1] exploited the linear dependency on rotational speed from gyroscopic influences of a flexible rotor system matrix to derive a gain scheduled $H_\infty$ controller that guaranteed a prescribed level of performance over a wide running speed range. Sivrioglu and Nonami [2] have also considered this technique and compared it with scheduled sliding mode control for a turbomolecular pump having a single active magnetic bearing. In these cases, a plant with dynamics that are dependent on time varying parameters was controlled using an algorithm that was also dependent on those parameters. However, an extension of this concept that has not been widely considered is the use of parameter dependent controllers where the parameters can be used to select the required performance and/or robustness of the closed loop system on-line. The plant itself need not be time varying. Such high order optimal controllers that are functionally dependent on a number of 'tuning' parameters are considered in this study.

## 2. CONTROL OF LINEAR PARAMETER DEPENDENT SYSTEMS

Control synthesis techniques for linear parameter dependent systems have been developed [3] based on a LMI formulation. Their main application has been to the design of gain-scheduled $H_\infty$ controllers for plants whose dynamics can be approximated by affine parameter dependent models. Provided these time varying parameters can be estimated or measured on-line, then the appropriate gain scheduled controller will ensure the $H_\infty$ performance criterion is satisfied over the entire range of considered parameter variations. Such synthesis techniques are applicable to plants that can be modelled with the state space equations:

$$\begin{aligned} \dot{x} &= \mathbf{A}(\boldsymbol{p})x + \mathbf{B}_1(\boldsymbol{p})\boldsymbol{w} + \mathbf{B}_2\,\boldsymbol{u} \\ \boldsymbol{z} &= \mathbf{C}_1(\boldsymbol{p})x + \mathbf{D}_{11}(\boldsymbol{p})\boldsymbol{w} + \mathbf{D}_{12}\,\boldsymbol{u} \\ \boldsymbol{y} &= \mathbf{C}_2\,x + \mathbf{D}_{12}\,\boldsymbol{w} + \mathbf{D}_{22}\,\boldsymbol{u} \end{aligned} \tag{1}$$

where the vectors $\boldsymbol{y}$ and $\boldsymbol{u}$ are the control outputs and inputs, $\boldsymbol{z}$ and $\boldsymbol{w}$ are the error outputs and disturbance inputs respectively, and $\boldsymbol{x}$ contain the dynamic states. The indicated state space matrices are affine functions of the vector of time varying parameters $\boldsymbol{p}(t)$, each element having defined lower and upper bounds $\underline{p}_i(t)$ and $\overline{p}_i(t)$:

$$\boldsymbol{p}(t) = [p_1(t),\dots, p_n(t)], \qquad \underline{p}_i(t) \le p_i(t) \le \overline{p}_i(t) \tag{2}$$

All parameter values will therefore be contained within an $n$ dimensional box having vertices $\boldsymbol{P}_{1\dots N}$. Defining the overall plant system matrix $S$ as

$$S(\boldsymbol{p}) = \left[\begin{array}{c|cc} \mathbf{A}(\boldsymbol{p}) & \mathbf{B}_1(\boldsymbol{p}) & \mathbf{B}_2 \\ \hline \mathbf{C}_1(\boldsymbol{p}) & \mathbf{D}_{11}(\boldsymbol{p}) & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{D}_{22} \end{array}\right] \tag{3}$$

then $S(\boldsymbol{p})$ can be formulated as a summation over the vertex system matrices $S(\boldsymbol{P}_j)$:

$$S(\boldsymbol{p}) = \alpha_1 S(\boldsymbol{P}_1) + \dots + \alpha_N S(\boldsymbol{P}_N) \tag{4}$$

where the coefficients $\alpha_i$ can be calculated from a linear decomposition of $\boldsymbol{p}(t)$:

$$\boldsymbol{p} = \alpha_1\,\boldsymbol{P}_1 + \dots + \alpha_N\,\boldsymbol{P}_N, \qquad \alpha_i \ge 0, \quad \sum_{i=1}^{N} \alpha_i = 1 \tag{5}$$

If the parameter dependent controller $S_c$ is formulated in the same way, as an interpolation of vertex controllers:

$$S_c(\boldsymbol{p}) = \alpha_1 S_c(\boldsymbol{P}_1) + \dots + \alpha_N S_c(\boldsymbol{P}_N) \tag{6}$$

where

$$S_c(\boldsymbol{p}) = \begin{bmatrix} \mathbf{A}_c(\boldsymbol{p}) & \mathbf{B}_c(\boldsymbol{p}) \\ \mathbf{C}_c(\boldsymbol{p}) & \mathbf{D}_c(\boldsymbol{p}) \end{bmatrix} \tag{7}$$

then the closed loop system matrix can be constructed as a summation over the vertex closed loop system matrices.

Existence conditions for the controller satisfying a closed loop $H_\infty$ specification ($\|T\|_\infty < \gamma$) require that two symmetric matrices $\mathbf{X}$ and $\mathbf{Y}$ can be found such that [4]:

$$\begin{pmatrix} \mathbf{L} & 0 \\ 0 & \mathbf{I} \end{pmatrix}^T \begin{pmatrix} \mathbf{A}_j\mathbf{X} + \mathbf{X}\mathbf{A}_j^T & \mathbf{X}\mathbf{C}_{1j}^T & \mathbf{B}_{1j} \\ \mathbf{C}_{1j}\mathbf{X} & -\gamma\mathbf{I} & \mathbf{D}_{11j} \\ \mathbf{B}_1^T & \mathbf{D}_{11j}^T & -\gamma\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{L} & 0 \\ 0 & \mathbf{I} \end{pmatrix} < 0$$

$$\begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{I} \end{pmatrix}^T \begin{pmatrix} \mathbf{A}_j^T\mathbf{Y} + \mathbf{Y}\mathbf{A}_j & \mathbf{Y}\mathbf{B}_{1j}^T & \mathbf{C}_{1j} \\ \mathbf{B}_{1j}\mathbf{Y} & -\gamma\mathbf{I} & \mathbf{D}_{11j} \\ \mathbf{C}_1^T & \mathbf{D}_{11j}^T & -\gamma\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{I} \end{pmatrix} < 0 \tag{8}$$

$$\begin{pmatrix} \mathbf{X} & \mathbf{I} \\ \mathbf{I} & \mathbf{Y} \end{pmatrix} \ge 0$$

for all $j$, where the index $j$ denotes the system matrices for each of the vertex parameter values. Matrices $\mathbf{L}$ and $\mathbf{M}$ are bases of the null spaces of $[\mathbf{B}_2^T\ \mathbf{D}_{12}^T]$ and $[\mathbf{C}_2\ \mathbf{D}_{21}]$ respectively.

If a common solution ($\mathbf{X}$ and $\mathbf{Y}$) can be found for all the vertex system matrix values then an arbitrary system matrix formed from a linear interpolation of the vertex system matrices (equations (3) and (4)) will also satisfy these inequalities and hence a parameter scheduled controller of the form given in equation (6) can be constructed [3].

## 3. GENERALISED LOOP SHAPING

The usefulness of the $H_\infty$ controller design method is enhanced when shaping (or weighting) filters are included in the plant formulation. These allow the closed loop frequency response of the system to be influenced through appropriate choice of shaping filters. Shaping filters are often chosen as diagonal transfer function matrices that act on the plant inputs and outputs. The most general case is shown in figure 1 and corresponds to the weighted open loop plant given in the Laplace transform domain as

$$\begin{bmatrix} \overline{Z}(s) \\ \overline{Y}(s) \end{bmatrix} = \begin{bmatrix} W_z(s)G_{zw}(s)W_w(s) & W_z(s)G_{zu}(s)W_u(s) \\ W_y(s)G_{yw}(s)W_w(s) & W_y(s)G_{yu}(s)W_u(s) \end{bmatrix} \begin{bmatrix} \overline{W}(s) \\ \overline{U}(s) \end{bmatrix} \tag{9}$$

A closed loop controller $K$ that satisfies an $H_\infty$ (peak gain) specification for the weighted plant can then be modified through combination with the shaping filters to give a controller $W_yKW_u$, that stabilises the unweighted plant. The closed loop response then satisfies

$$\|W_z T(G,K) W_w\|_\infty \le \gamma \tag{10}$$

The inclusion of shaping filters at the control inputs and outputs ($W_u$, $W_y$) can be used to ensure the controller has desired characteristics, such as integral action, or poles or zeros at specified locations.

The inequality equation (10) is equivalent to

$$\overline{\sigma}(W_z(j\omega))\overline{\sigma}(T(j\omega))\overline{\sigma}(W_w(j\omega)) \le \gamma \tag{11}$$

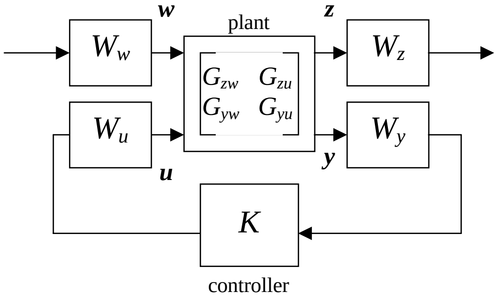


**Figure 1 Closed loop system with shaping filters**

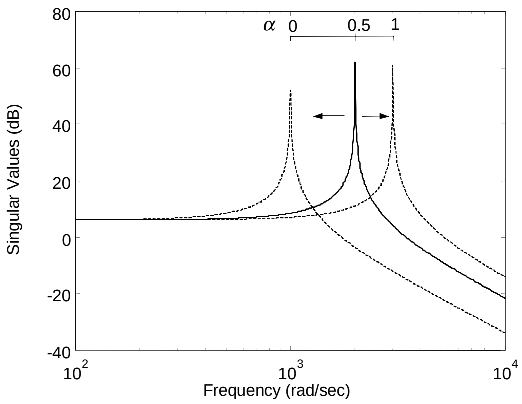


**Figure 2 Vertex shaping filters (- - -) and linear interpolation (——)**

The shaping filters ($W_w$, $W_z$) can therefore be chosen to place a frequency dependent bound on the maximum gain of the closed loop system frequency response. In most design problems, inclusion of $W_z$ alone is sufficient to give the required level of design influence.

## 4. LINEAR PARAMETER DEPENDENT SHAPING FUNCTIONS

The state space matrices for a shaping filter, if suitably chosen, can be given a linear dependence on certain characteristic parameters that are important in their application to loop shaping. Namely, the location of the poles and zeros and the overall gain, which are directly related to the shape of the filter. This is most easily demonstrated by considering the first order transfer function

$$F(s)=\frac{k(s-a)}{s-b} \tag{12}$$

having a pole at $s = b$, a zero at $s = a$ and a gain factor $k$. Consider the state space representation of this system given by

$$S_W(k,a,b)=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]=\left[\begin{array}{c|c} b & k \\ \hline b-a & k \end{array}\right] \tag{13}$$

It can be seen that with this representation the system matrices have an affine dependence on the specified parameters. Hence, the system matrix for a filter with arbitrary but bounded parameter values can be formed from a linear interpolation of vertex system matrices having the maximum and minimum parameter values. This is important as it allows the parameter dependent shaping filters to be specified within the type of control problem considered in section 2, for a linear parameter dependent plant. An affine dependence of the state space matrices on the poles and zeros can also be achieved for a second order transfer function with complex poles or zeros by using a real modal representation.

As an example, consider a second order transfer function with poles at $s=\pm j\omega_n$. If two such transfer functions with natural frequencies $\omega_1$ and $\omega_2$ are taken as vertex systems then the system matrices for a filter having a natural frequency $\omega_n$ where $\omega_1 < \omega_n < \omega_2$ can be formed from a linear interpolation of the corresponding vertex system matrices. This is illustrated in figure 2, which shows such a parameter dependent filter, the type of which will be used in subsequent controller design. The filter with natural frequency $\omega_n$ is constructed from the overall system matrix of two vertex filters with natural frequencies $\omega_1$ =1000 and $\omega_2$ = 3000 rad/s:

$$\begin{gathered} S(\omega_n)=\alpha\, S(\omega_2)+(1-\alpha)S(\omega_1) \\ \alpha=(\omega_n-\omega_1)/(\omega_2-\omega_1) \end{gathered} \tag{14}$$

Higher order filters can be constructed by connecting such first and second order filters in series, without losing the linear dependence of the system matrices on the pole and zero locations.

## 5. SYSTEM DESCRIPTION

The system model considered in this study is based on a turbomolecular pump having five control axes (although only transverse rotor motion and control will be considered). The magnetic bearings are of standard design with position feedback control being applied. The radial displacement of the rotor is measured, relative to the stator in two planes. External disturbance sources are present that can cause vibration of the system base, but also a direct forcing disturbance acts on the rotor, for example, due to unbalance forces. The running speed range of the rotor is 0-3000 rad/s. A schematic diagram is shown in figure 3.

The state space description of the system dynamics has the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= \mathbf{A}(\Omega)\boldsymbol{x}+\mathbf{B}_u\,\boldsymbol{u}+\mathbf{E}\,\boldsymbol{f} \\ \boldsymbol{y} &= \mathbf{C}_y\,\boldsymbol{x}-\boldsymbol{y}_b \end{aligned} \tag{15}$$

where $\boldsymbol{x}$ is the vector of rotor states (translatory and rotational displacements and velocities), $\boldsymbol{u}$ the vector of forces applied by the bearings, $\boldsymbol{f}$ the vector of direct forces acting on the rotor, $\boldsymbol{y}_b$ the vector of base displacements at the sensor locations and $\boldsymbol{y}$ the measured rotor displacement (relative to the base). The system matrices are derived from the rigid body dynamics of the spinning rotor and therefore the **A** matrix has a linear dependence on rotational frequency $\Omega$ through gyroscopic effects.

For magnetic bearings operating in an opposing pole pair configuration, with constant bias currents and differential driving mode, the linearised bearing force can be written as

$$\boldsymbol{u} = \boldsymbol{u}_c + K_z \, \boldsymbol{y}_m \qquad (16)$$

where $\boldsymbol{u}$ is the bearing control force due to the control current and $K_z$ is the negative stiffness of the bearing. If the rotor displacement at the bearings $\boldsymbol{y}_m$ is measured, then proportional feedback can be applied to give zero bearing stiffness. The transmitted force $\boldsymbol{u}$ is then equal to any additional control feedback applied and the base motion input is equivalent to a position demand signal.

For control synthesis, the system must be arranged to have the form of the open loop system given by equation (1). The exact constitution of the signals $\boldsymbol{w}$ and $\boldsymbol{z}$ will depend on the control problem being considered. In practice, $\boldsymbol{z}$ can be constructed from any combination of disturbance and reference signals acting on the system and $\boldsymbol{w}$ constructed from any combination of input signals and system states.

## 6. TUNEABLE CONTROLLER DESIGN

The concepts and methods explained in the previous sections can now be combined for the purpose of deriving controllers for which the dependent parameters can be used as tuning variables. There are many possibilities, with regard to the choice of shaping filters and tuning parameters. However, the general purpose of the control synthesis will be to obtain a number of vertex controllers that differ in a number of important indices, for example performance (in one or more sense), controller gain and/or robustness. From these vertex controllers, a parameter dependent controller (equation (6)) can be implemented that allows these properties to be smoothly varied on-line through an interpolation of the controller state space matrices.

It is beyond the scope of this study to investigate all the possibilities for such a design method, or to investigate rigorously the mathematical feasibility for any particular design formulation. Therefore, a number of examples will be chosen for demonstration and the potential for useful 'tuneability' resulting from the designs will be investigated.

### Design 1. Synchronous tracking vibration control

In some applications it is desirable that the orbit amplitude of the rotor due to unbalance excitation is minimised. Controller designs that have been proposed to achieve this include those using notch filters [5]. Alternatively, Matsumura *et al.* [6] used boundary constraints in an $H_\infty$ loop-shaping procedure to derive a controller having infinite gain at the appropriate running speed frequency. For the $H_\infty$ design method used here, the plant is formulated with disturbance inputs $\boldsymbol{w} = y_b$, and error outputs $\boldsymbol{z} = [\boldsymbol{y},\ \boldsymbol{y}+y_b]$. Vertex plants are selected, corresponding to two different rotational frequencies $\Omega_{1,2}$. A second order shaping filter $W_y$ acts on the output $\boldsymbol{y}$, having poles close to the imaginary axis with natural frequencies equal to the two rotational frequencies ($\omega_{1,2} = \Omega_{1,2}$). Solving the LMI problem (equation (8)) for the two vertex system matrices gives a parameter dependent solution for all rotational frequencies between the two vertex values. In effect the poles of the shaping filter, rather than being fixed, can track the disturbance frequency by being shifted up and down the imaginary axis of the complex plane via the linear parameter dependency demonstrated in section 4. This allows synchronous vibration to be minimised over a wide running speed range, or alternatively to be fine-tuned when running speed is fixed. An advantage of the method used here is that the running speed dependence of the plant **A** matrix can also be incorporated in the formulation and thereby give added guarantees of performance and stability over the running speed range considered.

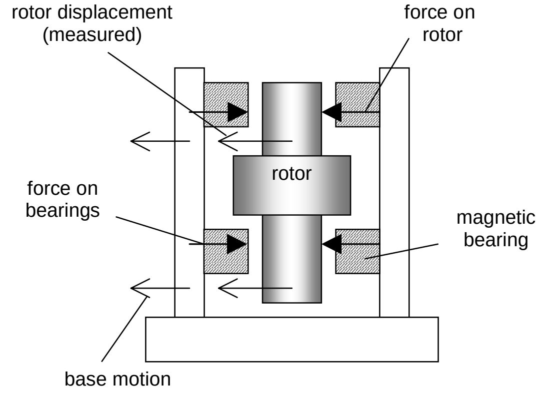


**Figure 3 Schematic diagram of experimental system**

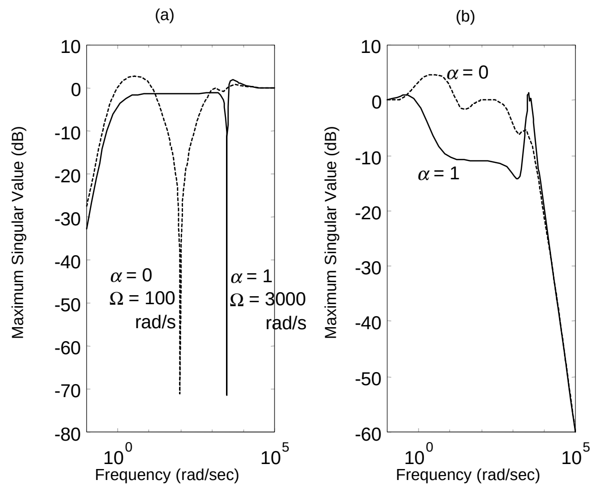


**Figure 4 Performance and robustness of the vertex systems for design 1: Maximum singular values of (a) sensitivity function (b) inverse sensitivity function**

The sensitivity function $(I+G_{yu}K)^{-1}$ and inverse sensitivity function $G_{yu}K(I+G_{yu}K)^{-1}$ for the two vertex system are shown in figure 4. Vertex shaping filters have been chosen having lightly damped poles with natural frequencies at 100 and 3000 rad/s and it can be seen that the vertex system sensitivity functions are minimised at these frequencies. A controller formed from a linear superposition of the two vertex controller matrices allows the sensitivity function to be minimised at synchronous frequencies within the interval 100-3000 rad/s (figure 5).

**Design 2. Tuneable sensitivity/robustness**

Loop shaping techniques allow a trade-off between achieving performance and robustness through minimisation of the sensitivity and inverse sensitivity functions over differing frequency bands. In rotor/magnetic bearing systems the desire is often to have a reduced sensitivity at low frequencies. For example, a small response to low frequency base motion disturbances is useful. At higher frequencies a low inverse sensitivity function is required for robustness to modelling errors or plant uncertainty, for example due to unconsidered flexural modes. A low inverse sensitivity function can also improve plant input disturbance rejection i.e. direct rotor forcing response. Often, the desired crossover frequency between achieving performance and robustness is uncertain, and the required level of robustness to plant uncertainties unknown. It is therefore advantageous to allow some tuning of these factors in the implementation of the controller and not just during the controller design stage.

In order to achieve this, consider the system having disturbance input $\boldsymbol{w} = \boldsymbol{y}_b$, and error outputs outputs $\boldsymbol{z} = [\boldsymbol{y}, \boldsymbol{u}]$. A parameter dependent shaping filter $W_c$ acts on the bearing force ($\boldsymbol{u}$) components of the output $\boldsymbol{z}$. The two vertex filters are shown in figure 6, together with an intermediate filter formed from a linear interpolation. Solving the $H_\infty$ control problem for the vertex systems gives the closed loop transfer functions shown in figure 7. It can be seen that these two closed loop systems have differing levels of performance and robustness. The two vertex controllers can therefore be used to form a tuneable controller, the tuning parameter of which can be used to select on-line the level of performance/robustness. It is envisaged that a number of tests would be performed on the system to deduce optimal selection of the tuning parameter. For example, the controller robustness could be reduced until high order flexural modes become destabilised, the robustness would then be increased slightly. Alternatively, tuning could be performed to optimise the trade-off between tracking performance and noise attenuation.

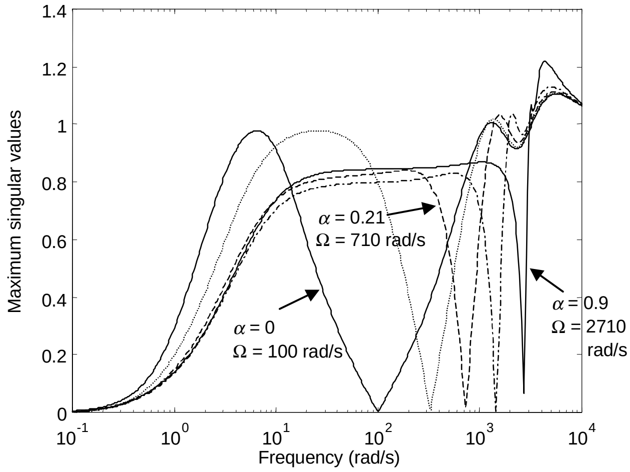


**Figure 5 Maximum singular values of sensitivity function for various rotational frequencies. Minima are collocated with rotational frequency**

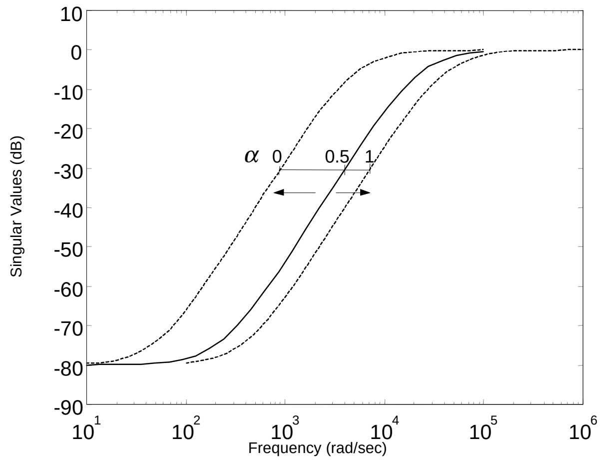


**Figure 6 Vertex shaping filters for design 2 $W_c$ (- - -) and linear interpolation (——)**

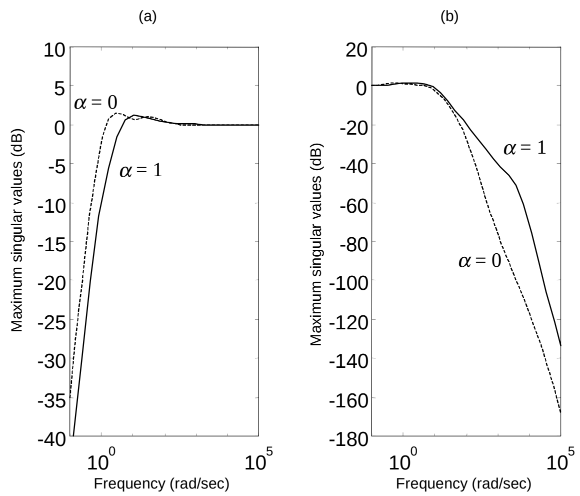


**Figure 7 Performance and robustness of vertex systems for design 2: Maximum singular values of (a) sensitivity function (b) inverse sensitivity function**

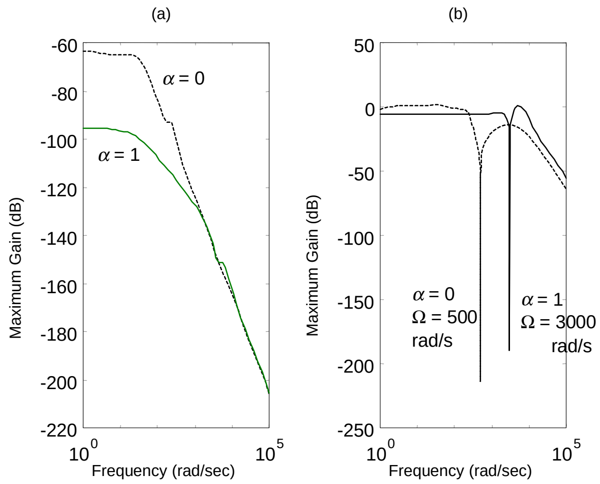


**Figure 8 Maximum RMS gain of the vertex systems for design 3 from rotor direct forcing input to (a) rotor displacements (b) bearing force**

### Design 3. Tuneable vibration transmission

In some applications it is desirable to have no control action at the running speed frequency. If this is achieved then the rotor will spin around its inertial axis. If the rotor unbalance is sufficiently small to give acceptable orbit sizes, then this technique has the advantage that orbit sizes are independent of running speed and also that bearing-rotor interaction forces are minimised. This has previously been achieved through the use of notch filters in series with low order controllers, with associated robustness problems in respect of destabilisation of rotor flexural modes. However, the technique developed here allows the use of notch filters in series with higher order robust controllers.

The formulation for this problem uses a filter at the plant control input $W_u$ having transmission zeros at the vertex synchronous frequencies. As in design 1, the transmission zeros can track the running speed frequency through the linear parameter dependency. Additionally, the running speed dependency can be incorporated in the plant system matrix. The maximum RMS response of the closed loop vertex systems to direct rotor forcing is shown in figure 8, together with the resulting bearing force magnitude.

## 7. CONCLUSIONS

In this paper a method has been proposed for deriving tuneable controllers. The examples used in this study, incorporated tuneability into the controller in order to achieve two differing types of objective. One type (designs 1 and 3) used parameter dependent weighting functions to give selected closed loop transfer functions that are minimised at the synchronous frequency. The linear parameter dependency of the plant and controller was exploited in order that this minimal frequency could track the rotational frequency of the rotor over a significant range. Although, these controllers could be viewed as tuneable, they could also be implemented as a self-scheduled controller if running speed is measured and used to automatically select the controller tuning parameter. The other type described, used a linear parameter dependent robustness-shaping filter to derive an optimal controller for which the performance and robustness levels could be tuned on-line through adjustment of a single parameter.

These studies suggest there is further scope for the use of controllers derived for linear parameter dependent systems, where the parameters are not treated as scheduling parameters (as conventionally has been the case) but are used as tuning parameters that can be manually selected on-line.

## ACKNOWLEDGEMENT

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