LINEAR-QUADRATIC OPTIMAL CONTROL OF ACTIVE MAGNETIC BEARINGS FOR HIGH SPEED ROTOR

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ABSTRACT

This paper describes an analytical solution to the Linear-Quadratic optımal design problem of the active magnetic bearing controller for a rigid high speed (gyroscopic) rotor. Dynamic properties of the optimal system are treated and compared with those obtained for the system with a conventional PD decentralised controller.

1. INTRODUCTION

The Linear-Quadratic (LQ) optimal control is widely used in the Active Magnetic Bearing (AMB) technology (Shweitzer, Bleuler and Traxler, 1994; Kim and Lee, 1994). The LQ-design problem is known to be based upon the solution of a non-linear matrix Riccati equation (Kwakernaak and Sivan, 1972). Numerical methods are usually used to solve this equation. But a numerical approach requires much design efforts and makes difficult utilization of the optimal control algorithm in real time calculations. Therefore, there is much practical interest to obtain an analytical solution of this problem. Such a solution in the case of One-Degree-Of-Freedom (1DOF, or second-order) AMB system may be found in (Hampton et al., 1992).

The tilting motions of a high speed rotor are coupled by gyroscopic forces and described by a fourth-order system. In the case of this system, where the control variables are magnetic forces and moments, the LQ-design problem has been analytically solved in (Zhuravlyov, 1991). The application of such a controller for the flywheel energy storage system prototype is described in (Zhuravlyov, Afanasiev and Lantto, 1994). In this paper the LQ-optimal controller is analytically designed in the case where the control variables are currents. The dynamic properties of the optimal closed-loop control system obtained (pole distribution and unbalance responses) are also treated analytically.

It should be mentioned that the optimal controller is not very simple to implement because it is multicoupled and speed-dependent; such a controller is known as centralized. Therefore, there is a quite natural tendency of designers to use more simple uncoupled and speed-independent controller in all applications including AMBs for high speed (gyro-

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scopic) rotors. Such a non-optimal decentralised controller is evidently characterised by greater control efforts. The question is: How much? This question is not new, it is analyzed by several authors (see Shweitzer, Bleuler and Traxler, 1994; Zhuravlyov, Afanasiev and Lantto, 1994), but their conclusions are different. For this reason, in this paper control efforts of the optimal AMB system and of the system with a conventional PD decentralised controller are compared.

2. MODELING OF ROTOR-BEARING SYSTEM

As shown in Fig.1, a rigid gyroscopic rotor of mass M, equatorial J_1 and axial J_3 principal moments of inertia spins at the constant rotational speed ω in two radial AMBs symmetrically located at the distance l from the center of mass C. The eccentricity $e = OC$ and the inclination γ of the principal axis of inertia characterize a static and dynamic unbalance of the rotor. We shall determine the position of rotor by coordinates x_0 and y_0 of the geometric center O and by angles of titling φ_x and φ_y about x and y axis, respectively. The two radial AMBs incorporate four pairs counteracting electromagnets with bias current i_0 and control currents i_1, i_2, i_3, i_4; the differential driving mode of the bearing electromagnets is assumed to be used.

The system model under consideration is given by

$$\begin{aligned}
&M\ddot{x}_0 - 2c_p x_0 = c_i(i_1 + i_3) + Me\omega^2 \cos\omega t \\
&M\ddot{y}_0 - 2c_p y_0 = c_i(i_2 + i_4) + Me\omega^2 \sin\omega t \\
&J_1\ddot{\varphi}_x + \omega J_3\dot{\varphi}_y - 2c_p l^2 \varphi_x = lc_i(i_2 - i_4) + (J_1 - J_3)\gamma\omega^2 \cos\omega t \\
&J_1\ddot{\varphi}_y - \omega J_3\dot{\varphi}_x - 2c_p l^2 \varphi_y = lc_i(i_3 - i_1) + (J_1 - J_3)\gamma\omega^2 \sin\omega t
\end{aligned} \tag{1}$$

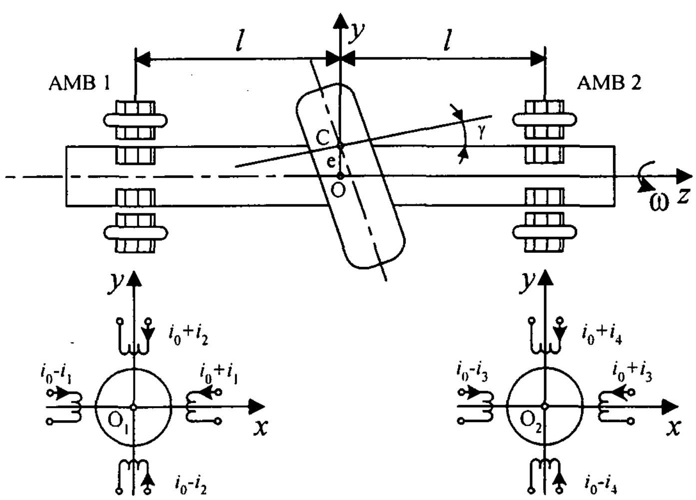

Figure 1. Model of a rigid gyroscopic rotor - AMB system

where c_p is the position "negative" stiffness, and c_i is the current stiffness of AMB. Note that the first two equations in (1) describe translational motions, and the others two coupled by gyroscopic tirms describe tilting motions of the rotor. We rewrite Eqs.(1) in the complex form

$$\ddot{z} - k_z^2 z = u_z + e\omega^2 \exp(j\omega t) \tag{2}$$

$$\ddot{\varphi} - jh\dot{\varphi} - k^2\varphi = u + (1-m)\gamma\omega^2 \exp(j\omega t) \tag{3}$$

where $j = \sqrt{-1}$, $z = x_0 + jy_0$, $\varphi = \varphi_x + j\varphi_y$, $u_z = c_i[(i_1 + i_3) + j(i_2 + i_4)]/M$, $u = lc_i[(i_2 - i_4) + j(i_3 - i_1)]/J_1$, $k_z = \sqrt{2c_p/M}$, $k = \sqrt{2c_p l^2/J_1}$, $m = J_3/J_1$, $h = m\omega$ is a gyroscopic parameter.

3. OPTIMAL AMB CONTROL

Consider a system modeled by the complex state-space description

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{4}$$

where $x(t)$ is the n complex state vector, $y(t)$ is the m complex vector of output variables, $u(t)$ is the m complex vector of control variables, and A, B and C are all constant complex matrices of appropriate dimensions. Consider the performance index

$$\int_0^\infty \left[y^*(t)y(t) + \rho\, u^*(t)u(t)\right]dt \tag{5}$$

where ρ is a positive weighting scalar, and the "*" denotes the conjugate transposition. The optimal control law minimizing index (5) is known to be given by

$$u(t) = -\rho^{-1}B^*Px(t) \tag{6}$$

where the $n \times n$ positive definite Hermitan matrix P is the solution of the complex valued algebraic matrix Riccati equation

$$C^*C + A^*P + PA - \rho^{-1}PBB^*P = 0 \tag{7}$$

Applying to system (2) the LQ-design procedure (4)-(7) yields the optimal Proportional-Derivative (PD) control law (see also Shweitzer, Bleuler and Traxler, 1994; Hampton et al., 1992)

$$i_1 + i_3 = -(g_1 x_0 + g_2 \dot{x}_0), \quad i_2 + i_4 = -(g_1 y_0 + g_2 \dot{y}_0) \tag{8}$$

where the feedback gains are given by

$$g_1 = M(\omega_0^2 + k_z^2)/c_i, \quad g_2 = M\sqrt{2(\omega_0^2 + k_z^2)}\,/c_i, \quad (\omega_0 \geq k_z) \tag{9}$$

Here ω_0 is the desired value of the undamped natural frequency of translational motions. The weighting scalar ρ correlates with ω_0 as $\rho = 1/(\omega_0^4 - k_z^4)$.

Applying to system (3) the LQ-design procedure (4)-(7), we have

$$\begin{gathered} x = (\varphi, \dot{\varphi})^T, \quad y = \varphi, \quad B = (0,1)^T, \quad C = (1,0), \\ A = \begin{bmatrix} 0 & 1 \\ k^2 & jh \end{bmatrix}, \quad P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + j\begin{bmatrix} 0 & p_4 \\ -p_4 & 0 \end{bmatrix} \end{gathered} \tag{10}$$

The second-order complex matrix Riccati Eq.(7) embodies four scalar equations

$$\begin{aligned} & p_2^2 + p_4^2 - 2\rho k^2 p_2 - \rho = 0 \\ & p_2 p_3 - \rho p_1 + h\rho p_4 - \rho k^2 p_3 = 0 \\ & p_3 p_4 - h\rho p_2 = 0 \\ & p_3^2 - 2\rho p_2 = 0 \end{aligned} \tag{11}$$

having the analytical solution

$$\begin{gathered} p_1 = \frac{4}{\rho^2 h^3} p_4^3 + \left(h - \frac{2k^2}{h}\right) p_4, \quad p_2 = \frac{2}{\rho h^2} p_4^2, \quad p_3 = \frac{2}{h} p_4, \\ p_4 = \left\{ \left[\frac{\rho^4 h^8}{64}\left(1 - \frac{4k^2}{h^2}\right)^2 + \frac{\rho^3 h^4}{4} \right]^{1/2} - \frac{\rho^2 h^4}{8}\left(1 - \frac{4k^2}{h^2}\right) \right\}^{1/2} \end{gathered} \tag{12}$$

The optimal control law is then

$$\begin{aligned} i_2 - i_4 &= -J_1\left[k_1(\omega)\varphi_x + k_2(\omega)\dot{\varphi}_x + k_3(\omega)\varphi_y\right]/\,lc_i \\ i_3 - i_1 &= -J_1\left[k_1(\omega)\varphi_y + k_2(\omega)\dot{\varphi}_y - k_3(\omega)\varphi_x\right]/\,lc_i \end{aligned} \tag{13}$$

where $k_1(\omega)$, $k_2(\omega)$ and $k_3(\omega)$ are, respectively, the optimal stiffness, damping and radial correcting factors of tilting motions of the rotor. These factors are given by

$$\begin{gathered} k_1(\omega) = \sqrt{\Omega_0^4 + h^4/16 - h^2k^2/2} + k^2 - h^2/4, \\ k_2(\omega) = \sqrt{2k_1(\omega)}, \quad k_3(\omega) = h\sqrt{k_1(\omega)/2} \\ (\Omega_0 \geq k; \quad h = m\omega; \quad m = J_3/J_1) \end{gathered} \tag{14}$$

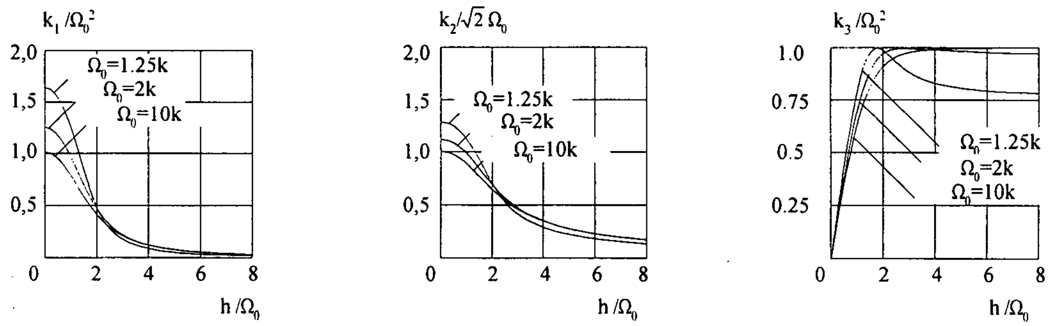

Figure 2. Variation of optimal stiffness (k_1), damping (k_2) and radial correcting (k_3) factors with rotational speed ($h = m\omega, m = J_3 / J_1$) for "expensive" control($\Omega_0 = 1.25k$), control of "intermediate cost" ($\Omega_0 = 2k$) and "cheap" control ($\Omega_0 = 10k$), where $k = \sqrt{2c_p l^2 / J_1}$ is the pole of open-loop system

Here Ω_0 is the desired value of the undamped natural frequency of tilting motions (about x and y axes) for the non-spinning rotor (i.e. with ω and h equal to zero). The weighting scalar ρ correlates with Ω_0 as $\rho = 1/(\Omega_0^4 - k^4)$. Fig.2 shows the variation of k_1, k_2 and k_3 with the gyroscopic parameter h. Functions $k_1(h)$, $k_2(h)$ and $k_3(h)$ have the following limits

$$\lim_{h\to\infty} k_1(h) = 2(\Omega_0^4 - k^4)/h^2 = 0, \quad \lim_{h\to\infty} k_2(h) = 2\sqrt{\Omega_0^4 - k^4}/h = 0,$$
$$\lim_{h\to\infty} k_3(h) = \sqrt{\Omega_0^4 - k^4} \tag{15}$$

It should be mentioned that the limit - case $\Omega_0 = k$ ($\rho \to \infty$) corresponds to minimum control efforts, it is usually called "expensive" control (we shall consider this case assuming $\Omega_0 = 1.25k$). The other limit - case $\Omega_0 >> k$ ($\rho \to 0$) corresponds to "cheap" control (we shall use value $\Omega_0 = 10k$).

The optimal control currents i_1, i_2, i_3 and i_4 can be easily determined from (8) and (13). It is evident that the optimal feedback gains are not constant; they vary with the rotational speed ω.

4. DYNAMIC ANALYSIS OF OPTIMAL SYSTEM

Translational motions of the rotor described by (2) and (8) are rather simple and, therefore, they are not treated in this paper.

Substituting (13) into (3) yields the optimal closed-loop control system for tilting motions of the rotor

$$\ddot{\varphi}+(k_2-jh)\dot{\varphi}+(k_1-k^2-jk_3)\varphi=(1-m)\gamma\omega^2\exp(j\omega t) \tag{16}$$

where k_1, k_2 and k_3 are dependent on ω in accordance with (14). Solving the complex characteristic equation of the system (16)

$$s^2+(k_2-jh)s+k_1-k^2-jk_3=0, \tag{17}$$

we find the optimal closed-loop poles

$$s_1=-\alpha+j\Omega_1,\quad s_2=-\alpha-j\Omega_2,\quad (\Omega_1\geq\Omega_2) \tag{18}$$

where

$$\alpha=\sqrt{k_1/2},\quad \Omega_{1,2}=\sqrt{k_1/2+h^2/4-k^2}\pm h/2 \tag{19}$$

Note, that Ω_1 is the nutation frequency; it increases with ω and tends to the value $\Omega_1=h$. The Ω_2 is the precession frequency; it tends to zero with ω. Evidently, the real (non-complex) closed-loop system has four poles: $s_{1,3}=-\alpha\pm j\Omega_1$ and $s_{2,4}=-\alpha\pm j\Omega_2$.

Given a particular solution of (16) as

$$\varphi(t)=\Phi_c^0\exp(j\omega t) \tag{20}$$

where Φ_c^0 is the complex amplitude, and substituting (20) into (16), we find the modulus of the amplitude

$$\Phi^0=\frac{|1-m|\gamma\omega^2}{\sqrt{\Omega_0^4+2(1-m)k^2\omega^2+(1-m)^2\omega^4}},\quad \left(\lim_{\omega\to\infty}\Phi^0=\gamma\right) \tag{21}$$

The complex control variable is given by

$$u=-\left[(k_1-jk_3)\varphi+k_2\dot{\varphi}\right] \tag{22}$$

Substituting (20) into (22) yields

$$u=U_c^0\exp(j\omega t), \tag{23}$$

where the complex amplitude U_c^0 has the modulus

$$U^0 = \Phi^0 \sqrt{k_1^2 + (\omega k_2 - k_3)^2}, \quad \left(\lim_{\omega\to\infty} U^0 = \gamma|2/m - 1|\sqrt{\Omega_0^4 - k^4}\right) \tag{24}$$

5. COMPARISON OF OPTIMAL AND DECENTRALISED CONTROL

Putting $\omega = 0$ into the optimal control law (13) yields the decentralised control law

$$u = -(f_1\varphi + f_2\dot{\varphi}) \tag{25}$$

where $f_1 = k_1(0) = \Omega_0^2 + k^2$ and $f_2 = k_2(0) = \sqrt{2(\Omega_0^2 + k^2)}$ are constant (speed-independent) feedback gains. The closed-loop system with the decentralised control is then

$$\ddot{\varphi} + (f_2 - jh)\dot{\varphi} + \Omega_0^2\varphi = (1 - m)\gamma\omega^2 \exp(j\omega t) \tag{26}$$

The amplitude of angle Φ and amplitude of control variable U of system (25) are given by

$$\Phi = \frac{|1 - m|\gamma\omega^2}{\sqrt{\left[\Omega_0^2 + (m-1)\omega^2\right]^2 + 2(\Omega_0^2 + k^2)\omega^2}} \tag{27}$$

$$U = \Phi\sqrt{(\Omega_0^2 + k^2)^2 + 2(\Omega_0^2 + k^2)\omega^2} \tag{28}$$

$$\lim_{\omega\to\infty}\Phi = \gamma, \quad \lim_{\omega\to\infty} U = \gamma\omega\sqrt{2(\Omega_0^2 + k^2)} \tag{29}$$

Figures 3 and 4 compares the typical responses to the step $\varphi_x(0) = 1$ of the rotor-AMB system with the optimal and decentralised controllers. It should be mentioned that in the case of non-spinning rotor $(\omega = 0, h = 0)$ the both controllers provide the same step responses; but in the case of high rotational speed step responses become essentially different. The optimal controller (Fig.3a) brings the axis of high-spinning rotor $(h = 10\Omega_0)$ to the zero state by the precession motion with the frequency Ω_2 during a half of period $2\pi/\Omega_2$; the nutation oscillations of high frequency Ω_1 and a small amplitude are superposed and damped very fast. The decentralised controller (Fig. 3b) is characterized by an evident oscillating transient process. Step responses of the system with optimal controller are provided by much less control efforts than with decentralised controller (Fig.4).

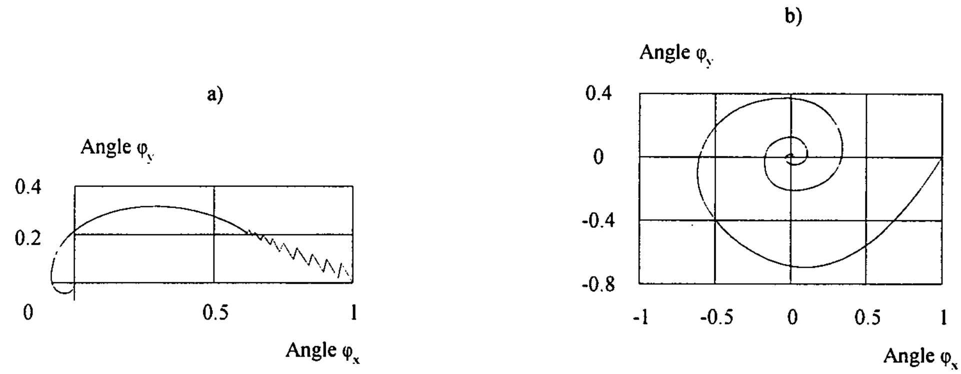

Figure 3. Step response (orbit view) of the system with optimal control (a) and decentralised control (b) for $\Omega_0 = 1.25k, h = 10\Omega_0$

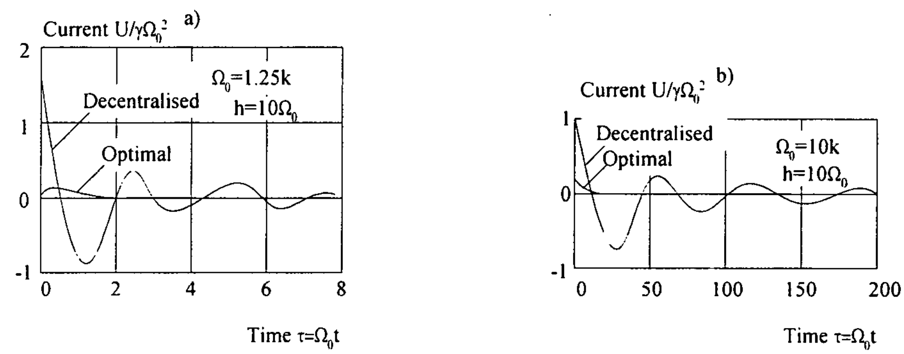

Figure 4. Step responses of the system with optimal control and decentralised control for "expensive" control (a) and "cheap" control (b)

Fig. 5 compares the unbalance responses of the rotor-AMB system with the optimal and the decentralised controllers. In all cases the optimal controller provides less control efforts than the decentralised controller. One can see that the saving in control efforts increases with the ratio of moments of inertia $m = J_3 / J_1$ and with the rotational speed. Note that value of m lies in the range $0 < m \leq 2$; the value $m = 2$ belongs to a thin disk with a massless shaft. The optimal controller is most advantageous for high speed (gyroscopic) rotors with the ratio $1 < m \leq 2$.

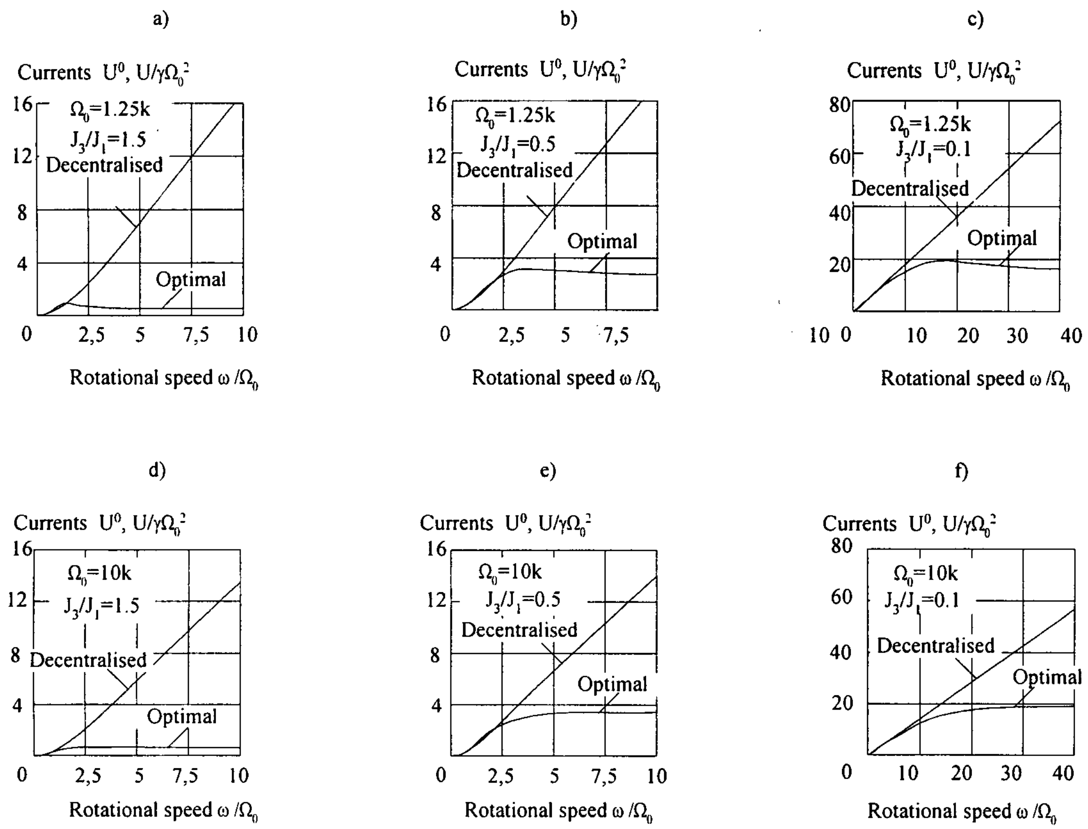

Figure 5. Unbalance responses of the rotor-AMB system with the optimal controller and decentralised controller for "expensive" control (a, b, c) and "cheap" control (d, e, f) and different ratio $m = J_3 / J_1$

6. CONCLUSIONS

This paper has described the analytical solution to the LQ-optimal design controller problem related to a current controlled AMB system for a rigid high speed (gyroscopic) rotor. This solution simplifies the controller design procedure and makes possible to use the optimal control law in real time calculations. It has been also shown that the significant savings in control efforts (compared with a conventional decentralised PD-controller) may be achieved by using the optimal controller, especially for rotors having an axial moment of inertia greater than an equatorial one.

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