# AMBIENT-TEMPERATURE PASSIVE MAGNETIC BEARINGS: THEORY AND DESIGN EQUATIONS

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## ABSTRACT

As described previously (R. F. Post, D. D. Ryutov, J. R. Smith, and L. S. Tung, Proc. of MAG '97 Industrial Conference on Magnetic Bearings, p. 167), research has been underway at the Lawrence Livermore National Laboratory on ambient-temperature passive magnetic bearings for a variety of possible applications. In the approach taken the limitations imposed by Earnshaw's theorem with respect to the stability of passive magnetic bearing systems are overcome by employing special combinations of elements, as follows: Levitating and restoring forces are provided by permanent-magnet elements that provide positive stiffnesses for selected displacements (i.e., those involving translations or angular displacement of the axis of rotation). As dictated by Earnshaw's theorem, bearing systems thus constructed will be statically unstable for at least one of the remaining possible displacements. Stabilization against this displacement is accomplished by using periodic arrays ("Halbach arrays") of permanent magnets to induce currents in close-packed inductively loaded circuits, thereby producing force derivatives stabilizing the system while in rotation. Disengaging mechanical elements stabilize the system when at rest and when below a low critical speed. The paper discusses theory and equations needed for the design of such systems.


## INTRODUCTION

There are many examples of rotating machinery, e.g., flywheel energy storage systems (electromechanical batteries) where it would be highly advantageous to employ "passive" magnetic bearing systems. Compared to "active" magnetic bearings (those using position sensors, electronic amplifiers, and control magnets) passive bearing systems could be less complex, less subject to failure, and, possibly, far lower in cost. Passive magnetic bearings must, however, overcome the consequences of Earnshaw's theorem (Earnshaw, 1839) This theorem asserts the impossibility of statically levitating systems employing only permanent magnets or electromagnets with fixed currents. One approach, pursued by Argonne National Laboratory (Weinberger, 1991) and by other groups, is to employ superconducting elements in the bearing system. Owing to their diamagnetic and other characteristics, superconductors evade Earnshaw's theorem. This solution, however, necessarily involves the use of cryogenic

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systems, with their attendant power requirements and complexity.

Research has been underway for some time at the Lawrence Livermore National Laboratory to build a theoretical and experimental base for designing ambient-temperature passive magnetic bearings for various applications (Post, et. al., 1997) In brief summary of the working principles undergirding this particular approach to passive magnetic bearing systems, they are the following:

(1) It is sufficient in the applications intended if stability is only achieved in the rotating state. That is to say, a centrifugally disengaging mechanical system can be used to insure stable support at rest (when Earnshaw's theorem applies).

(2) Stable levitation results if the vector sum of the force derivatives of the several elements of the bearing system, for axial, radial, and tilt-type displacements from equilibrium, is restoring. In this way it is possible to achieve Earnshaw-stable levitation using systems composed of multiple elements, no one of which is by itself stable against all of these displacements.

This article will present theoretical equations that we have developed to facilitate the design of Earnshaw-stable ambient-temperature passive bearing systems. Only brief comments will be made on the next level of stability-related problems encountered in rotating systems, rotor-dynamic instabilities. Our analyses of this latter problem (in the context of passive magnetic bearing systems), will be the subject of future papers.

## CRITERIA FOR EARNSHAW-STABILITY OF LEVITATED ROTORS

We first define criteria that, if met, will insure the Earnshaw-stability of a rotor supported by a passive bearing system. Figure 1 is a schematic drawing of such a system, in this case shown with the axis of rotation being vertical. The magnetic bearing components, A and B, shown above and below the rotor, may be composed of sub-elements, as described later.

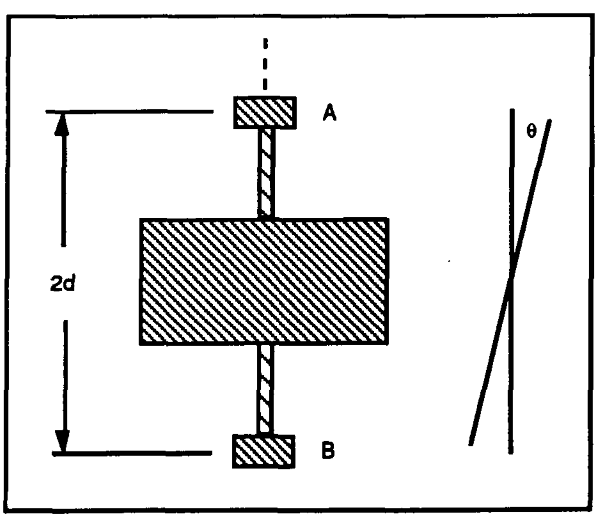


Figure 1

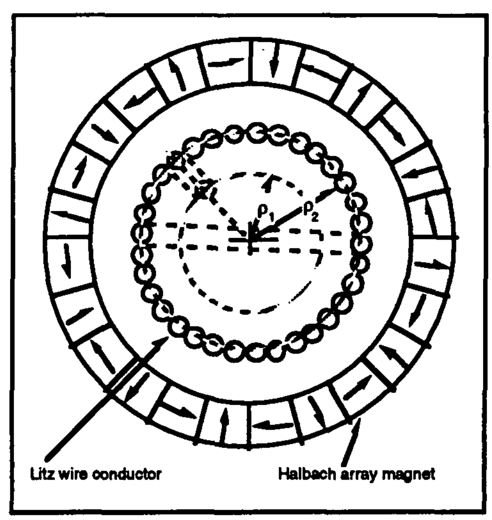


Figure 2

The combined characteristics of bearing components A and B are represented by stiffnesses $K_A$ and $K_B$ for lateral displacements (i.e., force derivatives with magnitudes $-K_A$ and $-K_B$). For axial displacements bearing components A and B will be characterized by stiffnesses $\beta_A K_A$ and $\beta_B K_B$. In the equations of motion of the rotor/bearing system we define transverse displacements of the center of mass by the variables x and y, and axial displacement by the variable z. Tilts of the axis can be characterized by two angles, $\theta_x$, $\theta_y$, representing tilts in the xz and yz planes, respectively. For small perturbations, lateral displacements of the axis in the bearings A and B can be represented as:

$$x_A = x + \theta_x d \qquad x_B = x - \theta_x d \qquad y_A = y + \theta_y d \qquad y_B = y - \theta_y d \tag{1}$$

Perturbations of the potential energy with respect to the equilibrium state can be written as:

$$U = \frac{K_A x_A^2}{2} + \frac{K_A y_A^2}{2} + \frac{K_B x_B^2}{2} + \frac{K_B y_B^2}{2} + \frac{(\beta_A K_A + \beta_B K_B) z^2}{2} \tag{2}$$

Using the Hamiltonian approach (Goldstein, 1950), and noting that the kinetic energy of the perturbations in the system, where we ignore gyroscopic effects, is a positive-definite quadratic form, one comes to the conclusion that the system is stable if and only if the potential energy (Eq. 2) is also a positive-definite quadratic form, i.e., if the following inequalities are satisfied (Goldstein, 1950):

$$K_A > 0, K_B > 0, \text{ and } (\beta_A K_A + \beta_B K_B) > 0 \tag{3}$$

Our task will be to find combinations of bearing elements satisfying these inequalities, recognizing that these conditions are more stringent than required in all situations.

In summary to this point, achieving Earnshaw-stability in a passive magnetic bearing system is an exercise in defining a bearing system composed of various elements, no subset of which is required to be stable against all displacements from equilibrium.

## AXIALLY SYMMETRIC PERMANENT-MAGNET ELEMENTS

To provide levitation and centering forces our bearing system utilizes axially symmetric bearing elements employing permanent magnet material. The simplest form of such elements are permanent magnets in the form of discs or annuli, magnetized in the axial direction and polarized so as either to attract or to repel. When facing each other, such elements in the attracting polarization provide radial centering, but are unstable against axial displacements, and vice-versa for the repelling polarization. Before presenting approximate analytical formulae for the forces and stiffnesses of such elements, we note a property of all axially symmetric elements. That is, the absolute value of the radial and axial stiffnesses of such elements for small displacements are in ratio of 1:2, while their signs are opposite. Thus, in calculating the axial stiffness of such an element one can be assured that the radial stiffness will be of opposite sign and of half the magnitude. A derivation of this result is given in the Appendix.

For the case of two equal-diameter magnetized discs with radius b (m.) and thickness h (m.), facing each other at a separation distance 2a (m.), where a < h and h << b, the magnitude of the axial force exerted by one disc on the other one is given approximately by Eq. (4) This equation was derived by integration of the force between two coplanar sheet currents, ignoring curvature effects, a valid assumption when a < h and h << b. (See the Appendix for a derivation of Eq. 4 and a discussion of the accuracy of the approximation used to derive it):

$$F_z = \frac{2B_r^2 bh}{\mu_0}\left\{(1+a/h)\ln[1+a/h]-(1+2a/h)\ln[\tfrac{1}{2}(1+2a/h)]+(a/h)\ln(a/h)\right\} \text{ Newtons} \quad (4)$$

Here $B_r$ (Tesla) is the remanent field of the permanent magnetic material (e.g., 1.4 Tesla for high-flux NdFeB magnet material), and $\mu_0 = 4\pi \times 10^{-7}$ hy/m. In the limit of zero separation between the magnets, the limiting axial force (for attracting magnets) becomes:

$$F_z(\max) = 2\ln(2)\left[\frac{B_r^2 bh}{\mu_0}\right] \text{ Newtons} \quad (5)$$

The axial force derivative (negative of the axial stiffness) is (for attracting magnets):

$$\frac{dF_z}{dz} = \frac{B_r^2 b}{\mu_0}\left\{2\ln\left[\tfrac{1}{2}(1+2a/h)\right] - \ln\left[(a/h)(1+a/h)\right]\right\} \text{ Newtons/meter} \quad (6)$$

For repelling magnets the magnitude of the force and of its axial derivative remain the same, but the signs of both change. Also, as noted, the magnitude of the force derivative for displacements transverse to the axis is half of that given by Eq. (6), and of opposite sign.

One deduces from the -2:1 or 2:-1 relationship between the axial and transverse force derivatives for repelling and attracting axially symmetric magnets that the best one can do with any combination of such elements is to achieve a neutrally stable state, i.e., one in which the force derivatives for both axial and transverse displacements exactly cancel at force equilibrium, a situation of no practical value. Using only such elements it is impossible to satisfy the criteria given by Inequalities (3) In his theses Basore (Basore, 1980, 1982) describes axially symmetric systems of such elements, interacting (upon displacement) with conducting elements. He concludes (op. cit., 1982) that for the type of system he considered the stabilizing effects (positive stiffnesses) he calculates for these interactions are too weak to be of practical value, so that for his systems active stabilization would be required. The passive stabilized systems that we will describe in what follows differ fundamentally from Basore's systems in that we utilize stabilizers employing periodic arrays of magnetic elements interacting with specially configured inductively loaded circuit elements, thereby achieving high positive stiffnesses with minimal parasitic losses in the undisplaced state.

Equations (4) and (6) are but examples of the types of equations needed for the design of the systems to be described. There are, of course, other axially symmetric magnet configurations that could be employed, such as concentrically nested annular magnets, and circular-pole magnets energized by permanent-magnet material. The relative stiffnesses and forces of these alternate configurations could be determined analytically, by the use of computer codes, or by measurements, and the data obtained plugged into the overall design to arrive at an Earnshaw-stable situation, employing the "stabilizer" elements to be described.

## HALBACH-ARRAY STABILIZERS

To achieve an Earnshaw-stable system employing axially symmetric permanent magnets it is necessary to add another ingredient. That is, one must introduce at least one element that will have a ratio of transverse to longitudinal force derivatives that deviates from the -2:1 or 2:-1 stiffness ratios of the axially symmetric elements, in such a way and of a sufficient magnitude that the bearing system taken as a whole can satisfy the requirements of Inequalities (3) or their generalizations (for example, in order to include gyroscopic effects).

The stabilizer elements to be described utilize periodic arrays of permanent magnets, configured in "Halbach arrays", named after the physicist who pioneered their analysis and use, Klaus Halbach (Halbach, 1985). These configurations, employing only permanent-magnet bars in their construction, represent optimally efficient ways to assemble such bars, creating a strong periodically varying magnetic field at one face of the array, while nearly canceling the field on the back face of the array. Devised for use in particle accelerators and free-electron lasers, they also turn out to be ideally suited for the stabilizers described here.

Halbach-array stabilizers take two geometrically different forms; "transverse" and "axial." Figure (2) is a schematic drawing of one form of a transverse stabilizer. A rotating Halbach array is shown surrounding a close-packed array of inductively loaded electrical circuits. The field from the rotating array produces a time-varying flux in each circuit. Above a low critical speed (determined by the circuit resistance and inductance), the induced current is phase shifted by nearly 90 degrees relative to the flux. This current, interacting with the rotating field, thereby provides a stabilizing force derivative for a system that would be otherwise unstable radially. The axial force derivative of such elements is very low, arising only from end effects, so that they do not contribute an appreciable destabilizing effect axially.

Since they employ non-axially symmetric fields, and since they involve dynamic (induction) effects, Halbach-array stabilizers are not subject to the constraints of Earnshaw's theorem. Thus either alone, or in combination with axially symmetric permanent-magnet elements, they enable the design of Earnshaw-stable systems (Post, 1996).

The magnetic field produced by a N-pole Halbach array as a function of radial position, $\rho < a$, and azimuthal angle, $\phi$, is given by the following equations (Halbach, 1985):

$$B_\rho = B_0 \left[\frac{\rho}{a}\right]^{N-1} \cos(N\phi), \qquad B_\phi = -B_0 \left[\frac{\rho}{a}\right]^{N-1} \sin(N\phi) \tag{7}$$

$$B_0 = B_r \left\{\frac{N}{N-1}\left[1 - \left(\frac{a}{b}\right)^{N-1}\right] \cos^N(\pi/M)\left[\frac{\sin(N\pi/M)}{(N\pi/M)}\right]\right\}, \quad N > 1 \tag{8}$$

In these expressions the quantity a (m.) is the inner radius of the Halbach array, and b (m.) is its outer radius. In Eq. (8) the quantity M is the total number of magnets in the array. In Figure 2, and for the type of array shown, there are 4 magnets per pole (i.e., 4 magnets per wavelength in the azimuthal direction), so that M = 4N., i.e. N = 6 in the figure.

We consider circuits for the windings of the stabilizer in the form of rectangular "window frames." In the simplest form the outer leg of this rectangular circuit, located at radius $\rho_2$, corresponds to one of the conductors shown in the figure, while the inner leg is located at radius $\rho_1 < \rho_2$, as shown on the figure. We will also later consider a case where the window frames span the diameter of the stator, the wires overlapping at the ends (also shown schematically in the figure).

The inductance of each circuit (self-inductance plus the effect of mutual inductance with adjacent circuits) is taken to be equal to $L_0$ (henrys), and its resistance is R (ohms). The conductor itself is litz wire. That is, it is composed of a multi-stranded bundle of fine strands of enamel-insulated copper wire. As later discussed, the use of litz wire greatly reduces the power losses associated with internal eddy currents in the wires.

The current induced in the circuits by the rotating Halbach array can be calculated from the flux produced by the array fields (Eqs. 7 - 8). At low speeds the current leads the flux by 90 degrees, resulting in drag forces but little repulsion. As the speed increases the phase lags until it approaches that of the peak flux, at which point the repelling force is maximal, and the drag torque is greatly reduced (varying inversely with the speed). The "transition speed," defined as the rotation speed where the repelling force has reached half its limiting value, is given by the relationship:

$$\omega_T = \frac{1}{N}\left[\frac{R}{L_0}\right] \quad \text{radians/sec.} \tag{9}$$

For typical stabilizers, this transition speed can be as low as a few hundred RPM. If it is desirable to further lower the transition speed, inductive loading can be added to each of the circuits (we have used small powder-core toroids for this purpose).

From the analysis the expression for the stiffness, $K_x$, of a stabilizer using the first type of windings, for displacements transverse to the axis of rotation, is:

$$K_x = \frac{(2N-1)\lambda M}{4\rho_2}\left[\frac{B_0^2 a h^2}{N L_0}\right]\left\{1 - \left(\frac{\rho_1}{\rho_2}\right)^N\right\}\left[\frac{\rho_2}{a}\right]^{2N-1} \quad \text{Newtons/m.} \tag{10}$$

The quantity $\lambda M$ corresponds to the total number of circuits, and the quantity h (m.) is the axial length of the Halbach array bars.

The analysis may also be extended to evaluate the ohmic power losses in the circuits relative to the stiffness. The expression derived is:

$$\frac{K_x}{P_0} = \frac{N(N-1)L_0}{2R\rho_2^2}\left\{\frac{1}{1 - \left(\frac{\rho_1}{\rho_2}\right)^N}\right\} \quad \text{Newtons m}^{-1}\text{ watt}^{-1} \tag{11}$$

When it is desirable to minimize the power losses associated with the stabilizer, a previously cited paper (Post, et. al., 1997) describes a version in which ohmic losses approach "zero" in the centered position. In this version, as noted earlier, the window-frame circuits have their legs on opposite sides of the stator, and the Halbach array has an even order N > 2 (N = 4, 6, 8 etc.). In this case there is flux cancellation in the centered condition and (except for residual currents arising from mechanical and magnetic tolerances associated with the Halbach array and the windings) the induced currents approach zero. For this case, the stiffness equation is:

$$K_x = \frac{4\pi}{\mu_0}\left\{\frac{B_0^2 h^2 N}{P}\right\}\left[\frac{c}{a}\right]^{2N-1} \quad \text{Newtons/meter} \qquad (12)$$

Here the quantity c (m.) is the radius of the cylindrical stator. The quantity, P (m.), is the perimeter of each circuit. This term arises from an evaluation of the mutual inductance of adjacent circuits, assuming no additional inductive loading is used. As a result, and if high-order Halbach arrays are used (N >> 4), the transverse stiffnesses attained can be high, in excess of $10^7$ Newtons/m.

The stabilizers just described address the problem of stabilization of a bearing system that is unstable transversely, but stable axially. An example would be a rotor levitated vertically between two sets of repelling magnets. For cases that are stable transversely (for example when attracting magnet pairs or their equivalent are used) and unstable axially, planar Halbach stabilizers arrays can be used. In such cases a planar circuit array is positioned midway between two planar Halbach arrays, mechanically coupled to each other at their inner radii so that they rotate together (illustrated schematically in Figure 3), and oriented azimuthally so that their axial fluxes cancel on the surface of a plane midway between them. Thus, when positioned midway between the Halbach arrays no currents are induced in the circuits, but currents and restoring forces arise from any axial displacement from this position. The design equation derived for this case is the following:

$$K_z = \frac{B_{0p}^2 N_c}{L_0}\left[bG(\alpha_b) - aG(\alpha_a)\right]^2 \quad \text{Newtons/meter} \qquad (13)$$

In this expression a (m.) is the inner radius of the planar Halbach array, b (m.) is its outer radius, the parameters $\alpha_a = Nh/a$ and $\alpha_b = Nh/b$, $N_c$ is the number of circuit wires, and the function $G(\alpha)$ is defined by the relationship:

$$G(\alpha) = \left[1 + \alpha\ln(\alpha) + \alpha(1 - C)\right], \; C = \text{Euler's const.} = 0.577.... \qquad (14)$$

The quantity $B_{0p}$ represents an effective mean value of the peak value of the magnetic field at the midplane between the two Halbach arrays at the inner radius, a. This quantity was calculated from results derived by Halbach (Halbach, 1985) for a linear array such as would be used in a "wiggler" in a free-electron laser. The result is given by (M = number of magnet bars in each array):

$$B_{0p} = 2B_r\left\{1 - \exp\left[-\frac{Nt}{a}\right]\right\}\left[\frac{\sin(\pi N/M)}{(\pi N/M)}\right]\exp(-Nh/a) \quad \text{Tesla} \qquad (15)$$

The use of planar Halbach-array stabilizers would permit the design of compact, Earnshaw-stable, passive bearing "cartridges." Such a cartridge would be composed of one planar stabilizer nested between two axially symmetric permanent-magnet bearing elements utilizing attracting pairs of magnets (positive stiffness transversely, negative stiffness axially). In such a cartridge the (unstable) axial forces of the two permanent-magnet pairs would buck each other, with the planar stabilizer providing axial centering to stabilize them. In the equilibrium position minimal power would be dissipated in the stabilizer, so that the parasitic losses of such a bearing cartridge could be very small. At the same time the cartridge as a whole would be relatively insensitive to temperature-induced changes in the magnetic field strength of the permanent-magnet pairs (assuming they were both at the same temperature), as the geometric symmetry of the arrangement would render the equilibrium position insensitive to such changes.

## EDDY-CURRENT LOSSES IN THE STABILIZER WINDINGS

In the stabilizers described here the legs of the circuits adjacent to the Halbach arrays are exposed to a rotating vector magnetic field, whether or not there is cancellation of the flux linked by these windings. This rotating field will induce residual eddy currents in the circuit wires, leading to losses. However, since eddy-current losses vary as the fourth power of the diameter of wire strands, the use of litz wire composed of many strands of fine wire can reduce these losses to the fractional-watt level. These losses in each litz wire strand are given by:

$$\frac{P_{ec}}{L} = \frac{\pi}{4}\left[\frac{B^2\omega^2a^4}{\rho_c}\right] \quad \text{Watts/meter of conductor} \qquad (16)$$

Here $\omega$ (rad./sec.) = $N\omega_0$, where $\omega_0$ is the angular frequency of the rotating system), $\omega$ is the frequency of the rotating field of magnitude B (Tesla), a (m.) is the radius of the conductor strand, and $\rho_c$ (ohm-meter) is its resistivity.

## COMMENTS ON THE STABILIZATION OF ROTOR-DYNAMIC INSTABILITIES

While the emphasis in this paper is on the design of Earnshaw-stable passive magnetic bearing systems, remarks are in order on the use of passive elements to stabilize rotor-dynamic modes. Two generically different techniques exist for stabilization: (1) eddy-current dampers, and, (2) anisotropic radial stiffness. The former involves, for example, stationary conducting sheets exposed to axially symmetric fields from rotating axially symmetric elements excited by permanent magnets, a technique employed, for example, by Fremerey (Fremerey, 1988). As derived in the previously cited paper (Post, et. al., 1997), stability against transverse whirl requires that the damping coefficient, $\beta$ (Newtons $m^{-1}$ sec.) should satisfy the equation:

$$\beta > \frac{\alpha}{\Omega_0}, \text{ stable,} \quad \Omega_0 = \sqrt{\frac{K}{M}} \text{ radians/sec.} \qquad (17)$$

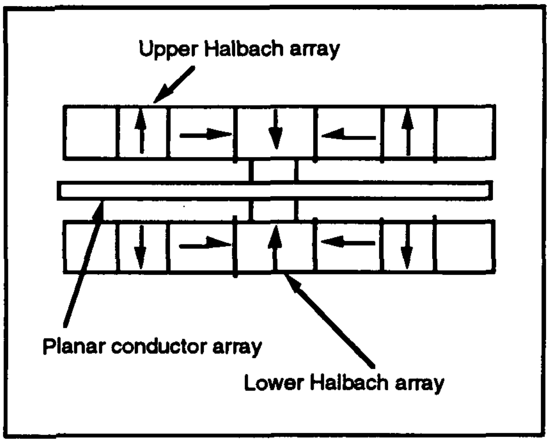


Figure 3

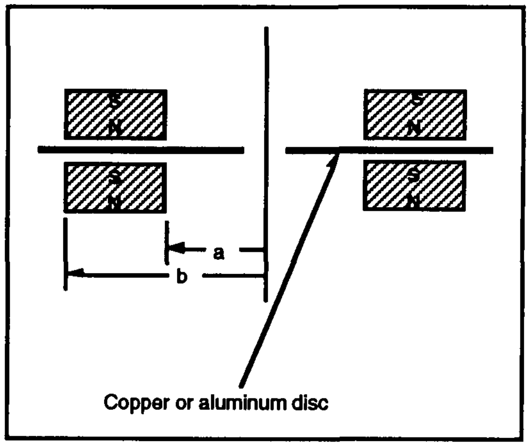


Figure 4

Here $\Omega_0$ is the natural frequency of transverse oscillations of the system for which the stiffness for transverse displacements is K (Newtons/meter) and the mass is M (kilograms). The parameter $\alpha$ (Newtons/meter) is the displacement-dependent drag coefficient (from whatever origin) that stimulates the transverse whirl.

The damping coefficient, $\beta$, for eddy-current dampers consisting of thin conducting discs positioned between two axially-symmetric annular poles, as illustrated in Figure 4, is given by a simple expression (derived in the Appendix). Subject to the restriction that the width of the poles, (b-a), and the thickness of the disc, t, satisfy the inequality $t(b-a) \ll \delta^2$ (Smythe, 1939), where $\delta$ (m.) is the skin-depth at $\Omega_0$, the equation is:

$$\beta = \frac{\pi t}{\rho}\int B_z^2 r dr \qquad \text{(Newtons m}^{-1}\text{ sec.)} \qquad (18)$$

Here $\rho$ (ohm-meters) is the resistivity of the material of the disc, and $B_z$ (Tesla) is the strength of the axial component of the magnetic field between the two poles.

If the two annular permanent magnets facing are each other have a gap, g, between them that is small compared to the radial width of the pole, (b-a), the magnetic field between the two poles is roughly constant within the gap, out to the edges of the poles. In this case Eq. (18) reduces to the simple approximate form:

$$\beta = \frac{\pi t}{2\rho}\left[B_z^2\,(b^2-a^2)\right] \qquad (19)$$

The second whirl-stabilizing technique, believed novel to our approach, is to introduce anisotropic stiffness in the radial stabilizers. This result can be accomplished by modulating the spacing or the inductive loading of the circuits as a function of azimuth, or by using a stator of elliptical cross-section. As described previously (Post, et. al., 1997), the degree of anisotropy required for this type of stabilization depends on the magnitude of the displacement-dependent drag terms, as follows:

$$\frac{K_x}{K_y} < \left[1 - \frac{2\sqrt{\alpha_x \alpha_y}}{K_y}\right] \text{, stable, } K_x < K_y \tag{20}$$

Here $\alpha_x$ and $\alpha_y$ (Newtons/meter), the displacement-dependent drag-force terms, have been assumed also to be anisotropic for generality. When these terms are small (as they are in many situations), the degree of anisotropy predicted to be required for stabilization is minimal. In addition to the stabilization introduced by anisotropy, displacement-dependent ohmic losses in the stabilizer windings can be expected to introduce some damping of transverse oscillations.

## EXAMPLE SYSTEM DESIGN

In this Section we outline the design of a vertical-axis system supporting a mass of 10 kilograms. ($F_z$ = 100 Newtons). The rotor will be located between two repelling magnets, so that the system is stable axially but unstable radially. Radial stabilization is then to be achieved by using upper and lower Halbach-array transverse stabilizers.

The parameters of the upper and lower repelling bearing sets are chosen by requiring that they both have the same (unstable) transverse stiffness, subject to the requirement that the difference in their axial forces should equal 100 Newtons (to provide levitation). We further assume that the ratio of the half-gap to the magnet thickness, i.e., the parameter (a/h), is the same for both magnet sets. It then remains to select the radius, b, and the relative thickness of the upper and lower magnet sets in order to satisfy the two requirements.

We first evaluate the equations for the axial force, Eq. (4), and Eq. (6) for the axial stiffness in the case that the parameter (a/h) is fixed at the value 0.1 (thus satisfying the small-parameter assumption with respect to (a/b) made in deriving the equations):

$$F_z\,(a/h = 0.1) = 0.97515\left[\frac{B_r^2 bh}{\mu_0}\right] \quad \text{Newtons} \tag{21}$$

$$K_z\,(a/h = 0.1) = 1.1856\left[\frac{B_r^2 b}{\mu_0}\right] \quad \text{Newtons/meter} \tag{22}$$

Recalling the 2:-1 ratio of stiffnesses for axially symmetric permanent-magnet elements that was discussed in Section III, we have for the transverse stiffness:

$$K_x\,(a/h = 0.1) = -\,0.5928\left[\frac{B_r^2 b}{\mu_0}\right] \quad \text{Newtons/meter} \tag{23}$$

We will also employ Eq. (5), defining the maximum (repelling) force, occurring as the gap, a, approaches zero. Putting in numerical values for the coefficient we have:

$$F_z\,(\max) = 1.3863\left[\frac{B_r^2 bh}{\mu_0}\right] \quad \text{Newtons} \tag{24}$$

We further assume that the relative magnet thickness of the lower magnet element (the one that provides the levitating force), is h/b = 0.2, and that the remanent field $B_r$ = 1.25 Tesla (standard-grade NdFeB material). For the upper magnet we will leave the thickness, h, as a variable to be determined. With these assumptions Eqs. (21), (23), and (24) become:

$$F_z\,(a/h = 0.1,\ h/b = 0.2,\ B_r = 1.25\ \mathrm{T}) = 2.4250 \times 10^5\, b^2 \quad \mathrm{N\ (lower\ mag.)} \tag{25}$$

$$F_z\,(\max) = 3.4474 \times 10^5\, b^2 \quad \mathrm{N\ (lower\ mag.)} \tag{26}$$

$$F_z\,(a/h = 0.1,\ B_r = 1.25\ \mathrm{T}) = 1.2125 \times 10^6\, bh \quad \mathrm{N\ \ (upper\ mag.)} \tag{27}$$

$$K_x\,(a/h = 0.1,\ B_r = 1.25\ \mathrm{T}) = -7.3709 \times 10^5\, b \quad \mathrm{Newtons/meter} \tag{28}$$

Imposing the requirement that the transverse stiffness of both the top and bottom bearing elements should be equal implies that both magnets should have the same radius (their thicknesses will not be equal, however). The common radius is determined by establishing a value for $F_z$ (max). To provide a robust levitating force we take this maximum value to be 400N (4 times the weight to be levitated). With this assumption one finds for the magnet radii, b = .03406 m. The thickness of the upper magnet element is determined by imposing the condition that the net levitating force should be 100 Newtons, yielding the equation:

$$2.4250 \times 10^5\, b^2 - 1.2125 \times 10^6\, b^2 (h/b) = 100\ \mathrm{Newtons} \tag{29}$$

Inserting the previously determined value of b we find (h/b) = 0.1289, for the upper magnet, to be compared to (h/b) = 0.2 for the lower magnet. Both of these values are consistent with the assumption made in the derivation of Eqs. (4) and (6), i.e., h << b.

To complete the design we need only determine parameters for radial stabilizers whose positive transverse stiffness sufficiently exceeds the negative stiffness of the levitator magnets to yield a desired net positive stiffness value. Inserting the value of b into Eq. (28) we find for the negative stiffness of each magnet set the value:

$$K_x\,(a/h = 0.1,\ B_r = 1.25\ \mathrm{T}) = -2.511 \times 10^4 \quad \mathrm{Newtons/meter} \tag{30}$$

If we employ stabilizers of the type represented by Eqs. (7),(8), and (12) with the parameters $B_r$ = 1.25 T, a/b = 0.8, h = 0.05 m. , P = 3h, N = 6, and c/a = 0.95, we obtain a positive stiffness value $K_x = 4.2 \times 10^5$ Newtons/meter, or about 16 times the negative stiffness of the levitating elements. It should be apparent that the stiffness of the Halbach array stabilizers needed to overcome the negative stiffnesses of the levitating bearings can readily be achieved, thus satisfying the Earnshaw-stability requirement, as given by inequalities 3.

## SUMMARY AND CONCLUSIONS

We have outlined the theory and presented design equations that can be used to perform the design of ambient-temperature passive magnetic systems that satisfy criteria for Earnshaw-stability. We have further sketched some approaches to the stabilization of rotor-dynamic instabilities in such systems. We have concluded the discussion by using the design equations to arrive at an example set of parameters for a vertical-axis system whose mass is 10 kilograms, finding reasonable values for all of the required parameters. The results presented are being incorporated in models that will explore practical issues that are sure to be encountered in converting the theoretical results into working systems.

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## APPENDIX

**(1) Proof of the 2:-1, and -2:1 ratio of stiffnesses of axially symmetric permanent-magnet bearing elements:**

Consider a circular current filament carrying current $j_0$, coaxially located in the axi-symmetric field of a permanent-magnet disc, $B_\rho$, $B_z$. We prove the hypothesis for this case. The general case follows by superposition. The stiffnesses are given by the equations ($\hat{\theta}$, $\hat{\rho}$, $\hat{z}$, $\hat{x}$ are unit vectors):

$$\frac{dF_z}{dz} = -K_z = \int_0^{2\pi} d\theta\, \hat{z}\cdot\nabla[(j_0\hat{\theta} \times B_\rho\hat{\rho})\cdot\hat{z}] = -\int_0^{2\pi} d\theta \left[ j_0 \frac{\partial B_\rho}{\partial z} \right] = -2\pi j_0 \frac{\partial B_\rho}{\partial z} \qquad \text{(A1)}$$

$$\frac{dF_x}{dx} = -K_x = \int_0^{2\pi} d\theta\, \hat{x}\cdot\nabla[(j_0\hat{\theta} \times B_z\hat{z})\cdot\hat{x}] = \int_0^{2\pi} \cos^2(\theta)d\theta \left[ j_0 \frac{\partial B_z}{\partial \rho} \right] = \pi j_0 \frac{\partial B_z}{\partial \rho} \qquad \text{(A2)}$$

The proof follows from the fact that $\nabla \times \mathbf{B} = 0$, i.e. $\frac{\partial B_z}{\partial \rho} = \frac{\partial B_\rho}{\partial z}$.

**(2) Derivation of the approximate equations for the force and stiffness of axially symmetric bearing elements consisting of two equal-diameter discs facing each other:**

The radius of the discs is b (m.), the gap distance is 2a (m.), and the thickness of each disc is h (m.). Curvature effects are neglected, implying that $a < h$ and $h << b$. The magnetization of each disc is represented by a surface amperian current, $j_0$ (amperes/m). In terms of the remanent field , $B_r$ (Tesla), $j_0 = B_r/\mu_0$. The force is obtained by integrating the force between two filamentary amperian surface currents over the vertical surface of the two magnets. The axial force per meter of circumference of the disc is given by the equation:

$$F_z = \frac{\mu_0 j_0^2}{2\pi} \int_a^{a+h} du \int_a^{a+h} dv \left[\frac{1}{u+v}\right] \text{ Newtons/meter} \qquad \text{(A3)}$$

Performing the integrations in Eq. (A3) and inserting the definition of the amperian surface current density in terms of the remanent field and of the circumference in terms of the

the radius, b (m.), there results the following approximate expression for the axial force between the two discs in the limit a < h and h << b:

$$F_z = \frac{2B_r^2 bh}{\mu_0}\left\{(1+a/h)\ln[1+a/h]-(1+2a/h)\ln[\tfrac{1}{2}(1+2a/h)]+(a/h)\ln(a/h)\right\} \text{ Newtons} \quad \text{(A4)}$$

Expressions for the limiting force as the gap approaches zero, and for the force derivative, $dF_z/dz$, are obtained from Eq. (A4) by passing to the limit a = 0 for the former and by differentiation with respect to 2a for the latter.

As a check on the accuracy of the approximation used in deriving Eq. A4, a comparison was made between a result obtained by utilizing the vector potential of a circular loop current to calculate the radial component of the magnetic field. From this field value the axial force exerted on an adjacent current loop can be evaluated by integration. For the case of two surface currents of vertical thickness, h = .001b, separated by a gap 2a = .02b, Eq. A4 was found to agree within about 1 percent with the "exact" result obtained using the vector potential.

To model amperian surface currents of much larger vertical height, h = 0.05b, separated by a larger gap, 2a = 0.05b, three loop currents were used to model each surface current, each loop carrying 1/3 of the total current. These loops were located at the top, middle, and bottom of the space occupied by each surface current in order to simulate its presence. In this case the agreement between the approximate formula and the "exact" one using the three-fold array of loop currents was within 5 percent.

The force equation that was derived from the vector potential of an amperian current loop and integrated numerically to obtain the above results is given by the expression:

$$F_z = \left[\frac{B_r^2 h^2}{2\mu_0}\right]\int_0^{2\pi}\frac{u\,d\theta}{\left[2(1-\cos\theta)+u^2\right]^{3/2}} \text{ Newtons} \quad \text{(A5)}$$

Here u = z/b, with z being the axial separation between the planes of the current loops.

**(3) Abbreviated derivation of the damping coefficient for eddy-current dampers:**

For a time-dependent transverse displacement (relative to the disc) of the facing poles (see Figure 4) of magnitude $\xi$ (m.), the varying field passing through the disc induces $\phi$ and r components of a divergence-free electric field satisfying the equation:

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi) - \frac{1}{r}\frac{\partial E_r}{\partial \phi} = \left[\frac{d\xi}{dt}\right]\frac{\partial B_z}{\partial r}\cos(\phi) \quad \text{(A6)}$$

The solution of this equation, subject to $\nabla\cdot \mathbf{E} = 0$, has the form

$$E_\phi = E_{\phi 0}(r)\cos(\phi); \quad E_r = E_{r0}(r)\sin(\phi) \quad \text{(A7)}$$

where the functions $E_{\phi0}(r)$ and $E_{r0}(r)$ satisfy the differential equations

$$\frac{1}{r}\frac{d}{dr}(rE_{\phi0}) - \frac{E_{r0}}{r} = \left[\frac{d\xi}{dt}\right]\frac{dB_z}{dr} \; ; \; \frac{d}{dr}(rE_{r0}) - E_{\phi0} = 0 \qquad \text{(A8)}$$

Eliminating $E_{r0}$ and solving the resulting equation for $E_{\phi0}$ one finds expressions for both these quantities in terms of integrals:

$$E_{r0} = \frac{(d\xi/dt)}{r^2}\int_0^r r_1 B_z(r_1)dr_1 \qquad E_{\phi0} = (d\xi/dt)B_z - E_{r0} \qquad \text{(A9)}$$

From these electric fields the dissipated power is found by integration by parts, and from the power (equal to friction force multiplied by velocity) the damping coefficient can be evaluated:

$$\beta = \frac{\pi t}{\rho}\int B_z^2 r dr \qquad \text{(A10)}$$

## REFERENCES

Basore, P. A. 1980. "Passive Stabilization of Flywheel Magnetic Bearings," M.S. Thesis, E.E. and Comp. Sci., M.I.T., Cam. MA

Basore, P. A. 1982. "Self-Stabilizing Magnetic Bearings for Flywheels," PhD Thesis, loc. cit.

Earnshaw, S. 1839. "On the Nature of the Molecular Forces which regulate the Constitution of the Luminiferous Ether," Trans. of the Cambridge Phil. Soc., VII. Part I, p. 97

Fremerey, J. K. 1988. "Radial Shear Force Permanent Magnet Bearing System with Zero-Power Axial Control and Passive Radial Damping," in Magnetic Bearings, Ed. G. Schweitzer, p. 25, Springer-Verlag

Goldstein, H. G. 1950. Classical Mechanics, Addison-Wesley

Halbach, K. 1985. "Application of Permanent Magnets in Accelerators and Electron Storage Rings," J. App. Phys. 57, 3605

Post, R. F. 1996. "Dynamically Stable Magnetic Suspension/Bearing System," U. S. Patent #5,495,221

Post, R. F. D. D. Ryutov, J. R. Smith, and L. S. Tung 1997. "Research on Ambient-Temperature Passive Magnetic Bearings at the Lawrence Livermore National Laboratory," Proc. of the MAG '97 Industrial Conf. and Exhibition on Magnetic Bearings, Aug. 21-22, 1997, p. 167

Smythe, W. R., 1939. Static and Dynamic Electricity, First Edition, McGraw-Hill, New York

Weinberger, B. R., et. al., 1991. "Low Friction in High Temperature Superconductor Bearings," App. Phys. Lett. 59, 1132