AN ELASTIC ROTOR MODEL FOR HIGH-SPEED ELECTRICAL MACHINES WITH ACTIVE MAGNETIC BEARINGS

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ABSTRACT
A program for modelling elastic rotor dynamics in active magnetic bearings has been made for PC-Matlab. The kinetic and potential energy integrals are diskretized and an equation of motion is obtained in a matrix form. Active magnetic bearing dynamics can be included in the matrix equation. From this matrix form the critical speeds, eigenfrequencies, mode shapes, unbalance responses and frequency responses can be calculated. A low order model for control system design purposes can also be calculated by modal analysis. The validity of the model is studied by comparing the calculated eigenfrequencies and eigenmodes with the analytic solutions of the Timoshenko beam equation. The program is also verified by testing real high speed rotors. The measurements are in a good agreement with the calculations.

INTRODUCTION
When designing high-speed machines an accurate prediction of the first critical speed of the rotor is needed, because in most cases we want to operate below the first critical speed. If, however, some critical speeds have to be passed, the corresponding eigen-modes have to be included in to the system model to ensure enough damping at critical speeds. When active magnetic bearings are used it is also necessary to ensure the stability of the first bending modes when the rotational speed is below the first critical speed. Otherwise, active magnetic bearings may cause troublesome ringing problems. Material damping will stabilize the eigenmodes which are far above the bandwidth of the active magnetic bearings.

Exact and reliable calculation of the eigenfrequencies and mode shapes of a complex rotor geometry demands three-dimensional finite element method. However, this method needs quite a lot of computing power, and using different programs in different tasks is trenuous. Besides, the rotor geometry used in high-speed technology is so simple that good results can be achieved by using a simpler method.

Our goal in the AMB design is to do all the modelling, control system design and analysis in a low cost personal computer. Matlab program with Simulink and different kinds of toolboxes provides an ideal environment for the AMB design. So, we decided to make also the programs for calculating the elastic rotor dynamics for Matlab. Actually, only a program that calculates a couple of matrixes from the rotor geometry had to be made. The rest of the operations, like eigenvalue and eigenvector computing is done with the powerful matrix handling functions of Matlab.

THEORY
By assuming that under deformations the rotor cross-sections remain planes, we can write the kinetic and potential energy of the rotor as integrals over the whole rotor length. Of course, if the rotor has large disks, this assumption of planes remaining planes does not hold, and results may be erroneous.

N diskretization points with six diskretization variables at each point are tied to the rotor centercurve. These variables define the shape of the rotor at the diskretization points. Between these points the rotor shape is approximated by third-order polynomials. The energy integrals can be calculated in terms of these variables and expressed in a matrix form. From these matrix forms the equation of motion is obtained according to Lagrangian mechanics.

$u(z)$ and $v(z)$ are the displacements of the rotor centre curve at position z in the directions of X- and Y-axis of an inertial frame. Because of the shear deformation, the tangent to the rotor centre curve differs from the normal of the rotor cross-section. $\alpha_X(z)$ and $\alpha_Y(z)$ define the rotation of the cross-section around the X and Y axis. $\phi_X(z)$ and $\phi_Y(z)$ are the shear angles.

$$\begin{aligned} \alpha_X &= -\frac{\partial v}{\partial z} + \phi_X \\ \alpha_Y &= \frac{\partial u}{\partial z} + \phi_Y \end{aligned} \tag{1}$$

The kinetic energy of the rotor is [1]

$$\begin{aligned} T = \frac{1}{2}\int_{z_0}^{z_1} \Bigg\{ & m\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right] + \\ & j_X\left[\left(\frac{\partial \alpha_X}{\partial t}\right)^2 + \left(\frac{\partial \alpha_Y}{\partial t}\right)^2\right] + \\ & \Omega j_Z\left[\alpha_Y \frac{\partial \alpha_X}{\partial t} - \alpha_X \frac{\partial \alpha_Y}{\partial t}\right]\Bigg\} dz + \\ & \frac{1}{2} J_Z \Omega^2 \end{aligned} \tag{2}$$

where z_0 and z_1 are the Z-coordinates of the rotor ends, m is the rotor mass per unit lengt, j_X is the inertial moment around X-axis per unit length (j_Y=j_X), j_Z is the inertial moment around Z-axis per unit length and J_Z is the total moment of inertia around the Z-axis. Ω is the rotational speed. The fourth row presents the kinetic energy with the rotation. If the rotational speed is supposed to be constant, this term is also constant and does not contribute to the equation of motion. The kinetic energy associated with the unbalance distribution is

$$T_U = -\Omega \int_{z_0}^{z_1} \begin{bmatrix} \sin(\beta) \\ \cos(\beta) \end{bmatrix}^T U \begin{bmatrix} \cos(\gamma) & \sin(\gamma) \\ \sin(\gamma) & -\cos(\gamma) \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} dz \tag{3}$$

where β is the rotor rotation angle ($\Omega = \dot{\beta}$), U is the unbalance per unit length and γ is the angle defining the direction of the unbalance. The potential energy associated with the deformation is [1]

$$\begin{aligned} V_D = \frac{1}{2}\int_{z_0}^{z_1} \Bigg\{ & EI\left[\left(\frac{\partial \alpha_X}{\partial z}\right)^2 + \left(\frac{\partial \alpha_Y}{\partial z}\right)^2\right] + \\ & \kappa AG\left[\phi_X^2 + \phi_Y^2\right]\Bigg\} dz \end{aligned} \tag{4}$$

where E is the modulus of elasticity (about 210 GPa for steel), and EI is the flexural rigidity of the beam ($I=\pi * r^4/4$ for a homogenous circular cross-section). A is the cross-sectional area, G is the shear modulus of elasticity (G=0,5*E/(1+ν) where Poisson's ratio $\nu \approx 0,3$ for steel). Slighly different values for the constant κ has been proposed in the litterature. I have used $\kappa \approx 0,9$. The gravitational potential energy is

$$V_G = \int_{z_0}^{z_1} mgv\, dz \tag{5}$$

The gravitational force is supposed to be in the direction of negative Y-axis.

High-speed rotors are relatively thick when compared with the rotor length. Therefore, the shear deformation is also included in the model. The solutions converge to the well known Timoshenko beam equation [2], when the number of diskretization points is increased. If the shear deformation is neglected (second row in (4)), but the rotational inertia of the rotor cross-sections is included, the solutions converge to the solutions of the Rayleigh beam equation. If also the rotational inertia is neglected (second line in (2)), the solutions converge to the solutions of the Euler beam equation. The Euler- and Rayleigh beam equations give slighly too high eigenfrequencies, and the error increases, when the diameter-length-ratio is increased. The relative error is bigger for higher order eigenfrequencies. In Figure 1 this error is drawn as a function of the diameter-length-ratio for a cylindrical rotor.

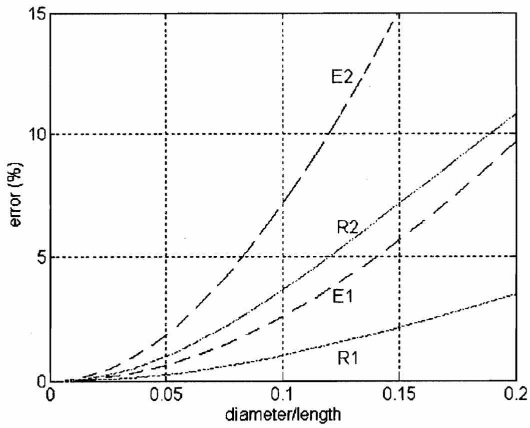

FIGURE 1: Errors in the first and second eigenfrequencies calculated from the Euler (E) and Rayleygh (R) beam equations when compared with the solution of the Timoshenko beam equation.

Althoug the eigenfrequencies vary quite a lot, the mode shapes are very similar. This is also true for more complex rotor geometries.
Now we tie N diskretization points to the rotor axis. The shape of the rotor is approximated by 6*N variables in the diskretization points. These variables are collected to a vector q

$$q = \left[u_1 \; v_1 \; \frac{\partial u}{\partial z}_1 \; \frac{\partial v}{\partial z}_1 \; \phi_{X1} \; \phi_{Y1} \; \cdots \; \phi_{YN} \right]^T \tag{6}$$

Between two successive diskretization points u and v are supposed to be third-order polynomials of z. Shear angles ϕ are supposed to vary linearly between the diskretization points. With these assumptions the integrals (2)...(5) can be calculated in terms of the vector q

$$\begin{aligned} T_q &= \frac{1}{2}\dot{q}^T M\dot{q} + \frac{1}{2}\Omega\dot{q}^T Gq \\ T_{Uq} &= -\Omega\left[sin(\beta)U_c^T - cos(\beta)U_s^T\right]\dot{q} \\ V_{Dq} &= \frac{1}{2}q^T Kq \\ V_{Gq} &= F_G^T q \end{aligned} \tag{7}$$

From these matrix forms the equation of motion is obtained according to Lagrangian mechanics

$$\frac{d}{dt}\frac{\partial(T_{tot} - V_{tot})}{\partial\dot{q}} - \frac{\partial(T_{tot} - V_{tot})}{\partial q} = F \tag{8}$$

where F is the generalized force in coordinates q. By substituting (7) into (8) we get the equation of motion for an elastic rotor:

$$\begin{aligned} M\ddot{q} + \Omega G\dot{q} + Kq = {} & B_F f \\ & + F_G \\ & + \Omega^2\left[sin(\beta)U_s + cos(\beta)U_c\right] \\ p = {} & C_p q \end{aligned} \tag{9}$$

Vector p contains the rotor displacements in the gap sensor positions and AMB reaction positions, because these positions are also needed to calculate the AMB reactions. f is a vector of forces. Matrixes M, G, K, B_F, F_G and C_p are calculated from the rotor geometry, the positions of the AMB reactions and positions of the gap sensors. Unbalance vectors U may be approximated by locating realistic unbalance point masses along the rotor. What is realistic is found out by experience. The material damping is neglected from this model. For steel the material damping is very low, but it should be kept in mind, because above the first critical speed the material damping may cause instability. The material damping may also stabilize a mode, which is predicted to be slighly unstable.
To equation (9) the AMB dynamics can easily be included in a state space form

$$\begin{aligned} \dot{x} &= A_{AMB}\,x + B_{AMB}\,p \\ f &= C_{AMB}\,x + D_{AMB}\,p \end{aligned} \tag{10}$$

Eqs. (9) and (10) may be combined in a straightforward manner to a same state space form system equation. A low order model for control system design and simulation purposes may be constructed by diagonalising (9) in the eigenvector base.

VERIFYING THE MODEL

The first test for the program was to compare the results with the analytic solutions of the Timoshenko beam equation. A 1 m long 0,2 m diameter cylinder was used. The mass density was 7800 kg/m^3, modulus of elasticity 210 GPa, and Poisson's ratio 0,3. The first four eigenfrequencies for a nonrotating shaft calculated analytically are 842 Hz, 2036 Hz, 3476 Hz and 5007 Hz. N equally distributed diskretization points were used, where N was varied. The last points were at the shaft ends. In Figure 2 the relative accuracy of the first four eigenfrequencies compared with analytic solutions is plotted as a function of the number of diskretization points.

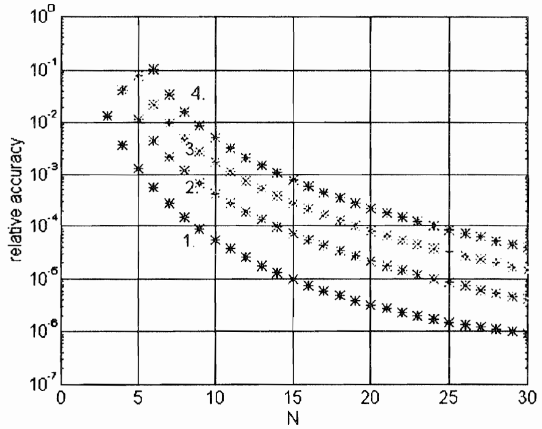

FIGURE 2: Relative accuracy as a function of the number of diskretization points.

The more eigenfrequencies we want to calculate, the more points we need.
Next test was a solid shaft specially made for testing these eigenfrequency calculations. The shaft geometry, and first two eigenmodes are plotted in Figure 3. The following material constants are used in the

calculations: E=210 GPa, ρ=7800 kg/m^3 and ν=0,3. The corner points of the rotor are listed in Table 1.

TABLE 1: Dimensions of the test shaft

Z-coordinate (mm)	radius (mm)
0	17
83	17
134	30,5
158	30,5
158	35
318	35
318	30,5
342	30,5
393	17
405,5	17

In Table 2 the first two eigenfrequencies of the shaft are calculated with different kinds of simplifications. Option 1- rotary inertia and shear deformation not included. Option 2- shear deformation not included and option 3- rotary inertia and shear deformation included. 15 equally distributed diskretization points were used.

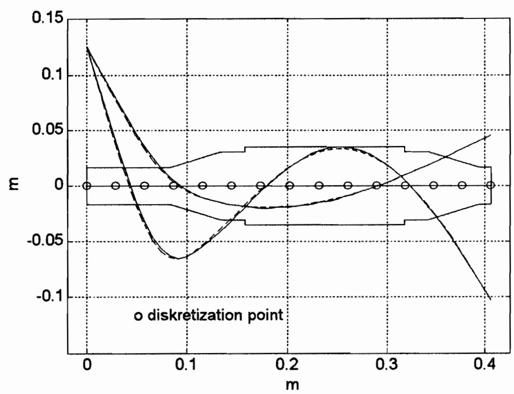

FIGURE 3: The geometry of the test shaft, and two first eigenmodes calculated with option 3 (solid line) and option 1 (dotted line).

TABLE 2: First two eigenfrequencies of the test shaft calculated with different kind of simplifications

measured	option 1	option 2	option 3
2081 Hz	2194 Hz (+5,4 %)	2147 Hz (+3,2 %)	2076 Hz (-0,2 %)
4225 Hz	4752 Hz (+12,4 %)	4485 Hz (+6,1 %)	4227 Hz (+0,06 %)

The diameter-length-ratio of the shaft is 0.17, so the results in Table 2 are in a good agreement with the Figure 1. Although the eigenfrequencies vary quite a lot, the mode shapes are almost similar. Figure 3 includes the mode shapes plotted with options 1 and 3. With real high-speed rotors the lamination sheets in the active magnetic bearings cause problems for eigenfrequency calculations. When the mass density of the laminations is supposed to be the same as for steel, and the modulus of elasticity is supposed to be zero, the results are close to the measured values. This is tested with a real high-speed rotor which geometry and three first eigenmodes are shown in Figure 4.

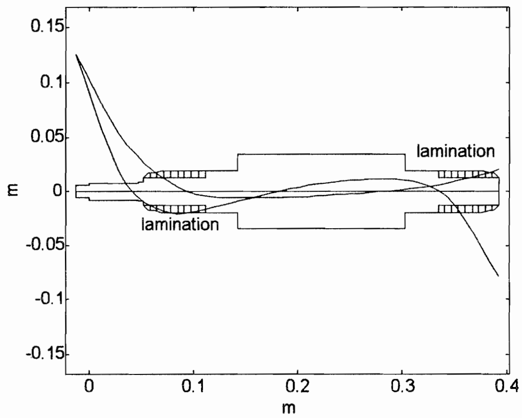

FIGURE 4: High-speed rotor with lamination sheets. Eigenmodes calculated supposing E_{lam}=0.

TABLE 3: Eigenfrequencies of a real high-speed rotor

measured	E_{lam}=0	E_{lam}=210 GPa
1679 Hz	1710 Hz (+1,8 %)	2217 Hz (+32 %)
2970 Hz	2886 Hz (-2,8 %)	3397 Hz (+14 %)
4125 Hz	4172 Hz (+1,1 %)	4888 Hz (+18 %)

All the eigenfrequencies above are calculated at zero rotational speed. Equation (9) predicts that the eigenmodes of the nonrotating rotor split in to a forward rotating mode (same direction as rotor) and a backward rotating mode. The eigenfrequency of the forward mode should increase, and the frequency of the backward mode should decrease. This phenomenon is measured with a 36 kW (shaft power), 50000 RPM air compressor. The machine has conical active magnetic bearings. The rotor geometry, and the first eigenmode are shown in Figure 5.
The measurement was done by supplying noise, the power of which was concentrated near 1000 Hz to

power amplifier of the AMB 2. A spectrum analysis by a digital spectrum analyzer was made for a displacement signal of the AMB 1. The two peaks in the power spectrum were very clear and sharp and were easily measurable. The results are plotted in Figure 6.

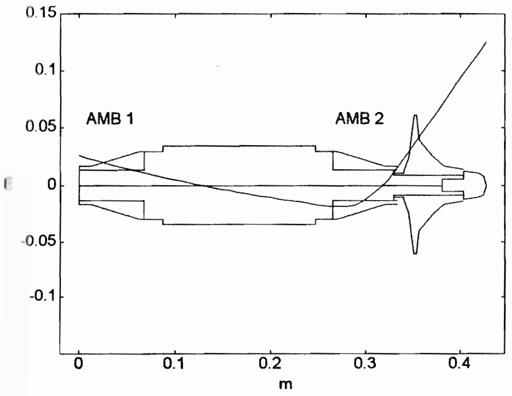

FIGURE 5: High-speed air compressor with conical active magnetic bearings

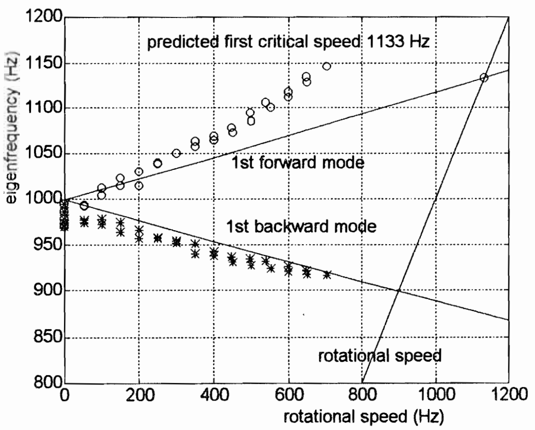

FIGURE 6: Splitting of the first eigenmode to the forward and backward modes. Calculated curves and measured points.

The calculations are made in free-free supports, because the effect of the AMB dynamics is negligible in this case. The eigenfrequencies of the 1st backward mode are very close to the calculated curve, whereas the measured eigenfrequencies of the forward mode are clearly higher than the calculated ones. This error may be due to modelling errors in compressor wheel. The eigenfrequency at zero speed varied about 30 Hz depending on the rotor temperature.

The calculation time was reasonable. After the rotor geometry of the air compressor was given, the calculation of the matrices *M*, *G* and *K* took about four minutes, when 15 diskretization points were used. After the matrixes had been calculated, the calculation of the eigenfrequencies and mode shapes took about half a minute, and the calculation of the curves at Figure 6 took about one minute. An 50 MHz 486 computer was used with Matlab version 4.0.

CONCLUSIONS

A kind of "one dimensional finite element method" is programmed to PC-Matlab to calculate elastic rotor dynamics. The measurements show that the accuracy of this method is high enough for simple rotor geometries, as high-speed rotors are. The computing time is also reasonable. The calculations show also that the shear deformation and rotary inertia of the rotor cross-sections should be taken into account.

REFERENCES

1. Han, R. P. S, Zu, J. W.-Z. 1993. Analytic dynamics of a spinning Timoshenko beam subjected to a moving load. Journal of the franklin institute, 330, 1, p.113-129.
2. Timoshenko, S., Young, D. H., Weawer, W. Jr. 1974. Vibration problems in engineering. 4. USA, John Wiley & sons, Inc. 521 p.