H^∞ Control of Flexible Rotor-Magnetic Bearing Systems

W.-M. CUI AND K. NONAMI

Abstract This paper deals with the flexible rotor-magnetic bearing system (FR-MBS) which has three concentrated masses. A mathematical model of the flexible rotor in the case of free-free condition is derived using finite element method. Then we derive the reduced order model for control system design by eliminating higher order modes of the mechanical and electrical-magnetic interaction system (full order model) beyond the first flexible mode. The H^∞ central controller is designed using the solution of H^∞ output feedback control. The control performance based on the mixed sensitivity problem is compared with it based on the robust stability. Simulations are done on the calculating model. The two unstable rigid modes can be easily controlled to be stable and the first flexible mode is better controlled than the uncontrolled case by the H^∞ controller. The spillover phenomena of the higher order modes do not generate. Also it is clear that the H^∞ control design has robustness to the variation of the parameters of the model.

1. Introduction The study of rotating machine supported by the magnetic bearings has been actively doing because of the excellent performance of it. Many papers have been reported about the active control and especially about both the vibration control and the stability for flexible rotors.

The flexible rotor has infinite vibration modes. When the flexible rotor is supported by the magnetic bearings by means of the conventional control methods, the spillover problem is very important. The H^∞ control is the powerful control theory to the model which has some uncertainty. The designs of robust stability control system[1] and vibration control system[2][3][4] based on the reduced order model have been proposed by using the H^∞ control theory. Also, several control designs of using the H^∞ control to the magnetic bearing systems[5][6][7] were presented. However, these studies deal with the control for the only rigid modes, and do not take into account the reduced order model.

This study is concerned with the H^∞ control design of the flexible rotor. Using a finite element method, the flexible rotor of continuous body is modelled as the discrete mass rotor[8]. Two reduced order models are considered here, one is the rotor which includes the only rigid modes, and the other includes both the rigid modes and the first order flexible mode. The H^∞

robust stability problem and the H^{∞} mixed sensitivity problem are used to control system design for the two reduced order models. From the results of the simulation, the control system designs based on the mixed sensitivity problem can not only cause no spillover but also control the each modes that the reduced order model includes to be stable and high damping. From the comparison of the control effects between the H^{∞} robust stability problem and the H^{∞} mixed sensitivity problem, the latter can both make the design of the controller which has integral feature in the design of controlling the rigid modes only and control the flexible mode to be high damping. Also, the control system of the H^{∞} control has a powerful robustness to the parameter variation.

2. Modelling of FR-MBS Figure 1 shows a radial type flexible rotor supported by the magnetic bearings at the both ends of the flexible rotor. We make the following assumptions:

(1) The attractive forces are proportional to the square of the coil current.
(2) Both electromagnetic bearings have the same characteristics.
(3) The induced voltages of electromagnets are ignored.
(4) The coil inductances are independent of frequency and gap length and are constant.
(5) This system is uncoupled between the x and y directions.
(6) Only small vibrations near equilibrium are considered.

And for simplicity, the analysis is done in the horizontal x direction. Under these conditions, we perform the modelling of the FR-MBS.

2.1 Modelling of flexible rotor Considering only the model of the flexible rotor in Fig.1(a), the equation of motion of the flexible rotor in free-free case is given by using a finite element method as follows:

$$\boldsymbol{M}_0 \ddot{\boldsymbol{q}} + \boldsymbol{K}_0 \boldsymbol{q} = 0 \tag{1}$$

where,

$$\boldsymbol{q} = \begin{bmatrix} x_1 & \theta_1 & x_2 & \theta_2 & x_3 & \theta_3 \end{bmatrix}^T$$

and M_0 is the mass matrix, K_0 the stiffness matrix. Figure 2 shows the mode shapes of this rotor.

2.2 Modelling of magnetic bearing system The following equation is obtained for one assembled electromagnet of the magnetic bearing system shown in Fig.1(b):

$$V = L\frac{dI}{dt} + R\,I \tag{2}$$

where V is the coil input voltage, L is the coil inductance, R is the coil resistance, and I is the coil current.

The attractive force of an electromagnet can generally be given by

$$P = \frac{\mu_0 A\, N^2 I^2}{H^2} \tag{3}$$

where P is the attractive force, μ_0 is the permeability, A is the face area, N is the number of winding turns, and H is the gap length. From the standpoint of small vibration near equilibrium, P, H, and I are given by

$$P = p_0 + p\,,\ \ I = i_0 + i\,,\ \ H = h_0 + h \tag{4}$$

where p_0 is the steady-state attractive force, i_0 is the steady-state current, h_0 is the steady-state gap length, p is the control attractive force, i is the control current, and h is the control gap length. Using the Taylor series expansion for small values of i and h and assuming $i_0 >> i$ and $h_0 >> h$, we can finally get the following attractive control force in linear terms:

$$p = 2p_0\left(\frac{i}{i_0} - \frac{h}{h_0}\right) = 2p_0\frac{i}{i_0} - 2p_0\frac{h}{h_0}. \qquad (5)$$

where the first term is the control attractive force, the second the bias attractive force.

2.3 Modelling of FR-MBS In Fig.1 the flexible rotor as Eq.(1) is restricted by the attractive forces of Eq.(5) as follows:

$$\boldsymbol{M}_0\ddot{\boldsymbol{q}} + \boldsymbol{K}_0\boldsymbol{q} = \boldsymbol{F}\,\boldsymbol{p} \qquad (6)$$

where,

$$\boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \qquad \boldsymbol{p} = \begin{bmatrix} p_l \\ p_r \end{bmatrix}$$

$$p_l = -4p_0\left(\frac{i_l}{i_0} - \frac{x_1}{h_0}\right): \text{forces of the } AMB\text{-}Ls$$

$$p_r = -4p_0\left(\frac{i_r}{i_0} - \frac{x_3}{h_0}\right): \text{forces of the } AMB\text{-}Rs.$$

The bias attractive forces and the control attractive forces of Eq.(6) are separated as follows:

$$\boldsymbol{M}_0\ddot{\boldsymbol{q}} + \boldsymbol{K}\,\boldsymbol{q} = \boldsymbol{F}_i\,\boldsymbol{i} \qquad (7)$$

where,

$$\boldsymbol{i} = [\ i_l \quad i_r\ ]^T \qquad \boldsymbol{K} = \boldsymbol{K}_0 + \boldsymbol{K}_i: \ \boldsymbol{K}_i = diag\left(-4\frac{p_0}{h_0}, 0, 0, 0, -4\frac{p_0}{h_0}, 0\right)$$

$$\boldsymbol{F}_i = \begin{bmatrix} -4\frac{p_0}{i_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4\frac{p_0}{i_0} & 0 \end{bmatrix}^T$$

By using the method of the mode analysis, we choose the following normalized modal matrix,

$$\boldsymbol{q} = \boldsymbol{\Phi}\,\xi \qquad (8)$$

Equation (7) is transformed to the form in modal coordinate as follows:

$$\ddot{\xi} + \Lambda\,\dot{\xi} + \Omega^2\,\xi = f_i\,\boldsymbol{i} \qquad (9)$$

where

$$\boldsymbol{I} = \boldsymbol{\Phi}^T\boldsymbol{M}\,\boldsymbol{\Phi} \qquad \Omega^2 = \boldsymbol{\Phi}^T\boldsymbol{K}\,\boldsymbol{\Phi} \qquad \Lambda = 2\,\zeta\,\Omega \qquad f_i = \boldsymbol{\Phi}^T\,\boldsymbol{F}_i$$

Moreover, considering the relation between the steady state voltage and the steady state current of Eq.(2), the state equation of the electromagnetic-mechanical system is given by

$$\dot{\boldsymbol{x}}_f = \boldsymbol{A}_f\,\boldsymbol{x}_f + \boldsymbol{B}_f\,\boldsymbol{u} \qquad (10)$$

where

$$\boldsymbol{x}_f = \begin{bmatrix} \dot{\xi} & \xi & \boldsymbol{i} \end{bmatrix}^T \qquad \boldsymbol{u} = [\ v_l \quad v_r\ ]^T$$

$$\boldsymbol{A}_f = \begin{bmatrix} 0 & \boldsymbol{I} & 0 \\ -\Omega^2 & -\Lambda & f_i \\ 0 & 0 & \boldsymbol{E}_i \end{bmatrix} \qquad \boldsymbol{B}_f = \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{E}_v \end{bmatrix}$$

$$\boldsymbol{E}_i = \begin{bmatrix} -\frac{R}{L} & 0 \\ 0 & -\frac{R}{L} \end{bmatrix} \qquad \boldsymbol{E}_v = \begin{bmatrix} -\frac{1}{L} & 0 \\ 0 & -\frac{1}{L} \end{bmatrix}$$

If the rotor displacements at the magnetic bearings can be measured, the output equation is

$$\boldsymbol{y} = \boldsymbol{C}_f\,\boldsymbol{x}_f = [\ x_1 \quad x_3\ ]^T \qquad (11)$$

2-4 Reduced order model of FR-MBS Generally the flexible rotor has infinite vibration modes in the case of free-free support. If we divide into the rotor to two parts using finite element method, the discrete model with six order modes is obtained. Because the FR-MBS is originally unstable, we must control it to be stable. In this case, there are only two unstable rigid modes, and the flexible modes are essentially stable. Here, the construction of the reduced order model is considered upon the standpoint to stabilize the two rigid modes and to control the vibration of flexible modes.

The reduced order model is constructed by truncation of the higher order modes in modal coordinates. Here, the state equation and the output equation including till the *kth* order mode are as follows:

$$\begin{aligned} \dot{\boldsymbol{x}}_r &= \boldsymbol{A}_r \boldsymbol{x}_r + \boldsymbol{B}_r \boldsymbol{u} + \boldsymbol{D}_r \boldsymbol{w} \\ \boldsymbol{y} &= \boldsymbol{C}_r \boldsymbol{x}_r \end{aligned} \tag{12}$$

Concerning the reduced order model of Eq.(12), the control system designs are done in two cases.

a. the case which only the rigid modes are controlled ($k = 2$).

b. the case which the rigid modes and the first flexible mode are controlled ($k = 3$).

In addition, the closed loop system has to maintain the robust stability without spillover caused by higher order modes ignored in both cases.

3. H^{∞} control system design H^{∞} control is the control method of making a estimating function, which is defined in frequency domain, to be minimum (or smaller than some value γ). The estimating function is written as the following H^{∞} norm,

$$\| G(s) \|_{\infty} = sup\ \sigma \left\{ G(j\omega) \right\} \tag{13}$$

Here, $\sigma \left\{ G(j\omega) \right\}$ is the maximum singular value of $\boldsymbol{G}(j\omega)$, i.e. the root of the eigenvalue of $\boldsymbol{G}^*(j\omega)\,\boldsymbol{G}(j\omega)$ ($\boldsymbol{G}^*(j\omega)$ is conjugate transpose of $\boldsymbol{G}(j\omega)$). Considering the H^{∞} control system design of the reduced order model of Eq.(12), the block diagram of the closed loop system with the H^{∞} controller is shown in Fig.3.

Here, $\boldsymbol{A}_r$, $\boldsymbol{B}_r$, $\boldsymbol{C}_r$ are the reduced order control objects, and $\boldsymbol{u}$ is the control force, $\boldsymbol{y}$ is the actual observation output, z_1 and z_2 are control values. $\boldsymbol{W}_1(s)$ and $\boldsymbol{W}_2(s)$ are weighting function matrices. And $\boldsymbol{H}(s)$ is the following H^{∞} controller,

$$\boldsymbol{H}(s) = \boldsymbol{C}_H (s\boldsymbol{I} - \boldsymbol{A}_H)^{-1} \boldsymbol{B}_H \tag{14}$$

We design the H^{∞} control system based on the reduced order model, namely designing the H^{∞} controller that can not only cause no spillover with the closed loop system of both the reduced order model and the actual full order model (robust stability), but also control the vibration modes included in the reduced order model (high damping).

In this study, we carry out the control system design for the FR-MBS using two methods, that is, the H^{∞} robust stability problem and the H^{∞} mixed sensitivity problem. The features of control results will be made clear comparing with the two methods.

3-1 Control system design of only rigid modes If $k = 2$, the reduced order model of Eq.(12) becomes the control system design model including only rigid modes.

a. Control system design of robust stability problem Robust stability problem is to design a robust controller for the model uncertainty which is equivalent to the model difference between the actual plant of the full order model and the design model of the reduced order model.

We get the block diagram of the robust stability problem when $W_2(s) = 0$ in Fig.3. Now let's consider the following equation:

$$T(s) = H(s)\,[\,I - P_r(s)\,H(s)\,]^{-1}\,P_r(s) \tag{15}$$

where

$$P_r(s) = C_r\,(\,s\,I - A_r\,)^{-1}\,B_r \tag{16}$$

$T(s)$ is called the complementary sensitivity function. It is used to estimate the degree of the robust stability of the closed loop system to the model error of control object. The essential condition of the robust stability is to minimize the H^∞ norm of the complementary sensitivity function. When there is some modelling error, the robust stability condition is defined as the following equation based on the small gain theorem,

$$\|\,\Delta(s)\,T(s)\,\|_\infty < 1 \tag{17}$$

We assume that the uncertainty $\Delta(s)$ is defined using some weighting function as follows:

$$\bar{\sigma}\left\{\,\Delta(j\omega)\,\right\} \le W_1(s) \tag{18}$$

The condition that the closed loop system with uncertainty should be stable is given by following expressions,

i) H^∞ controller $H(s)$ makes the $P_r(s)$ to be stable.

ii) $\|\,W_1(s)\,T(s)\,\|_\infty < 1$ (19).

b. Control system design of mixed sensitivity problem[3] In Fig.3, considering the sensitivity reduction ($W_2(s) \neq 0$), we obtain the formulation of the mixed sensitivity problem as follows:

$$\left\|\begin{array}{c} W_1(s)\,T(s) \\ W_2(s)\,M(s) \end{array}\right\|_\infty < 1 \tag{20}$$

where,

$$M(s) = [\,I - P_r(s)\,H(s)\,]^{-1}\,P_r(s) \tag{21}$$

is called the settling function. If the settling function is small in lower frequency domain, it is well known that the poles of poor damped modes can move into left side on the complex plane, and the regulator with good response can be designed.

3-2 Control system design model including till the first flexible mode The reduced order model of Eq.(12) becomes the model including both the rigid modes and the first flexible mode if $k = 3$. The formulation of control system design is just the same with section 3-1.

4. Simulation Table 1 shows the parameters used in the simulation model. Simulations are done in two ways: the control system design of only the rigid modes and that of till first flexible mode included.

4-1 Simulations of 3-1 The weighting functions for control only the rigid modes in the case of the flexible rotor is shown in Fig.4. The specification of the control is to stabilize the unstable rigid modes. That is to levitate by assuming the flexible rotor like the rigid rotor. There are two methods for stabilization by using H^∞ control to the rigid rotor, the controls of the robust stability problem and the mixed sensitivity problem. Table 2 shows the poles of the closed loop system obtained by controls of both the robust stability problem and the mixed sensitivity problem. It is found that the two unstable rigid modes are stabilized in both control cases, and the spillover caused by the truncation of flexible modes does not occur. Also, the control effects of these two cases are almost the same.

Table 3 shows the poles of the controller. Figure 5 shows the frequency characteristics of the controller designed based on the robust stability problem, and Fig.6 shows that designed based on the mixed sensitivity problem. Comparing Fig.5 with Fig.6, the characteristics are very different from each other in the frequency domain smaller than 1rad/sec. That means it has integral action in the frequency domain from 0.01 to 1rad/sec in Fig.6. It is necessary to keep powerful stiffness called static stiffness to various disturbances in the very lower frequency domain. The characteristics of the conventional *PID* controller is almost the same as that of Fig.6, it is desirable to design the control system of magnetic bearing system using the mixed sensitivity problem from the point of this meaning. Of course, we have to say that the conventional *PID* controller can't take into account spillover problem.

4-2 Simulations of 3-2 Figure 7 shows the weighting functions of the model including till first flexible mode. Table 4 shows the poles of the closed loop system obtained by the control of both the robust stability problem and the mixed sensitivity problem. It is clear that although the two unstable rigid modes can be stabilized in the case of control of robust stability problem and the first flexible mode can not be controlled. Using the band-pass type weighting function $W_2(s)$, the damping ratio of the first flexible mode is controlled from 1/1000 in the case without control to 1/27. The vibration of the first flexible mode is well controlled. Table 5 shows the poles of the controller.

Figure 8 shows the unbalance responses of the flexible rotor having $0.5g\cdot mm$ unbalance at the center disk of the rotor.

Figure 9 shows the responses of the initial displacement at lift off. We can see that it can be levitated with good stability in the case of the flexible rotor. The initial values of each displacement are $x_1 = 0.001m$, $x_2 = x_1 = 0$.

4-3 Robustness to parameter variations It has been made clear that the robust stability to spillover caused by the truncation of higher modes is sufficiently guaranteed by the results of above mentioned. That means it is useful to use H^∞ control to the control system design for the FR-MBS.

Concerning the actual machines, it is also the important problem that the characteristic of the control system gets to be worse with parameter variations. Figure 10 shows the responses of the initial displacement in the case that the mass of the center disk increases twice. Figure 11 shows the responses of the initial displacement in the case that the bias attractive forces varies to 50% of the design value. Comparing Fig.10 and Fig.11 with Fig.9, we can find out that H^∞ control has powerful robustness to the parameter variations because the characteristics of the control system are almost unchangeable.

5. Conclusions Simulations are carried out by using the two methods of robust stability problem and mixed sensitivity problem of H^∞ control theory for the FR-MBS. The conclusions is summarized as follows:

(1) The H^∞ control system to levitate with the good stability had been designed using

the reduced order model of the FR-MBS.

(2) As to the reduced order model including only rigid modes, the controller designed based on the H^{∞} robust stability problem has not the integral feature.

(3) As to the same model, the controller designed based on H^{∞} mixed sensitivity problem had the same powerful integral feature as that of the conventional *PID* controller.

(4) As to the model including till the first flexible mode, it is necessary to design the controller based on the H^{∞} mixed sensitivity problem for the sake of vibration control of the flexible mode.

(5) With the view of robustness to the spillover and the parameter variations, the H^{∞} control method is useful to the control system design of the FR-MBS.

References

(1) Nonami, K., Wang, J.W., and Yamazaki, S., 1991. "Spillover Control of Magnetic Levitation Systems Using H^{∞} Control Theory." Transactions of the Japan Society of Mechanical Engineers, Ser.C, 57(534): 568-575.

(2) Nonami, K., Wang, J.W., Sampei, M., and Mita, T., "Active Vibration Control of a Flexible Rotor Using H^{∞} Control Theory." The 1991 ASME Design Technical Conferences-13th Biennial Conference on Mechanical Vibration and Noise, September 22-25, 1991, Miami, Florida, DE-Vol.35,pp.85-92.

(3) Cui, W.M., Nonami, K., and Nishimura, H., 1991. "Active Vibration Control of Multi-Degree-of-Freedom Systems by H^{∞} Optimal Control Theory." The Second Symposium on Motion and Vibration Control., Kawasaki, Japan, pp.189-195.

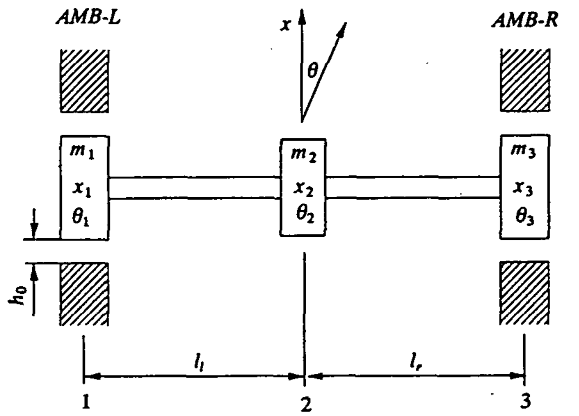
(4) Nonami, K., Nishimura, H., and Cui, W.M., 1991. "Active Vibration Control of Multi-Degree-of-Freedom Systems with Active Dynamic vibration Absorber Using H^{∞} Control Theory." The Second Symposium on Motion and Vibration Control., Kawasaki, Japan, pp.196-202.

(5) Fujita, M., Shimizu, M., and Matsumura, F., 1989. "On the Robust Control of a Magnetically suspended Flexible Beam System." The 1st Robust Control Research Meeting., Sapporo, Japan, pp.125-134.

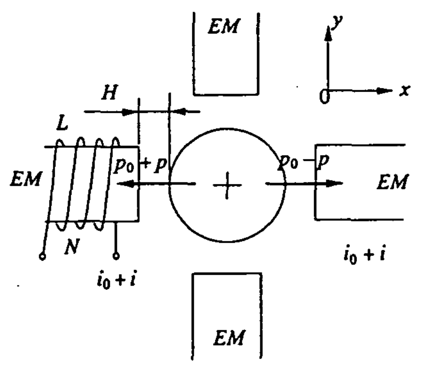
(6) Miura, A., Ikeda, K., and Kimura H., 1991. "Robust Control of a Magnetic Levitation System." The 20th SICE Symposium on Control Theory., Kariya, Japan, pp.265-269.

(7) Hatake, K., Fujita, M., and Matsumura, F., 1991. "Trial of Magnetic Bearing System Using H^{∞} Control Theory." The 3rd Electro-magnetics Symposium Proceedings, Kiryu, Gunma, Japan, pp.54-59.

(8) Nonami, K., Yamanaka, T., and Tominaga M., 1990. "Vibration and Control of a Flexible Rotor Supported by Magnetic Bearings." JSME International Journal,Ser.III, 33(4): pp.475-482.



(a) Flexible rotor model



(b) Model of magnetic bearing unit

Fig.1 Model of flexilbe rotor-magnetic bearing system

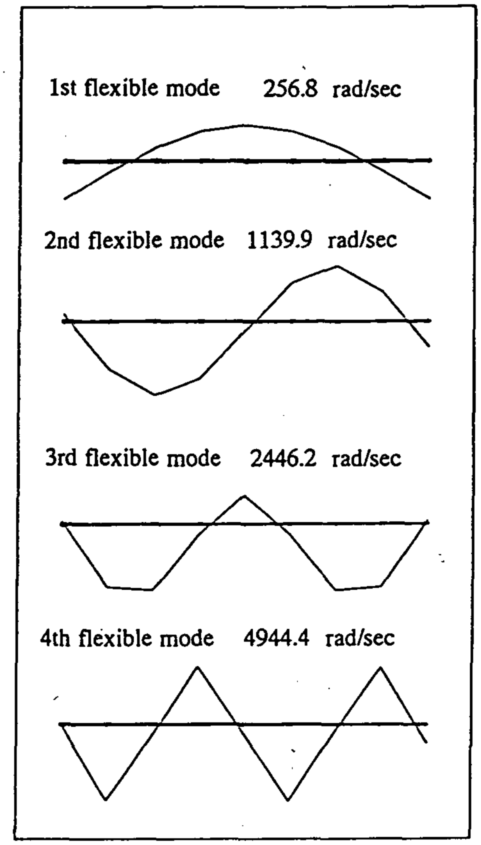


Fig.2 Mode shapes of flexible rotor

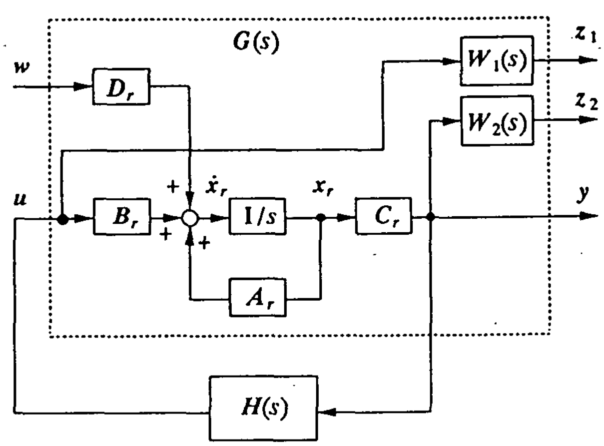


Fig.3 Block diagram of closed loop system with H^∞ controller
(for reduced order and generalized system)

Table 1. Specification of simulation model

Parameter		Value	Unit
Mass	m_1	1.50	kg
	m_2	0.80	kg
	m_3	1.50	kg
Length	$l_l = l_r$	0.50	m
Diameter	d	0.02	m
Damping ratio	ζ_i $(i=1,\cdots,4)$	0.001	
Gap	h_0	0.001	m
Bias current	i_0	1.0	A
Bias attractive force	p_0	50.0	N
Coil inductance	L	0.33	H
Coil resistance	R	23.6	Ω
Frequency of flexible modes	ω_1	40.9	Hz
	ω_2	181.5	Hz
	ω_3	389.5	Hz
	ω_4	787.3	Hz

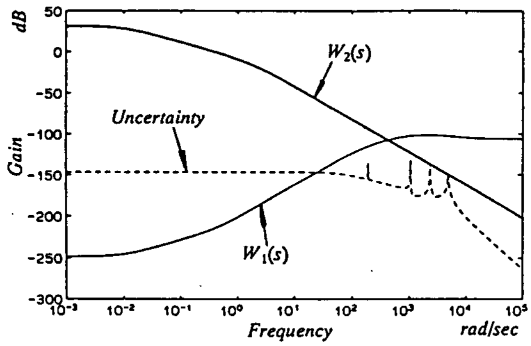


Fig.4 Singular values of system uncertainty and weighting functions

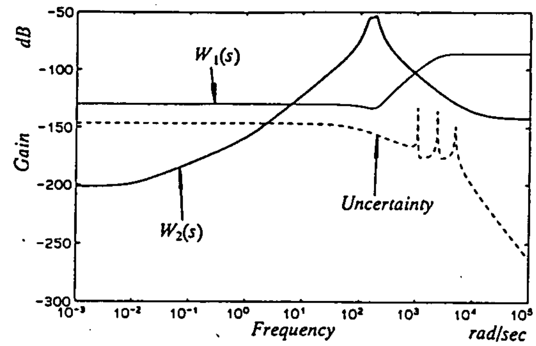


Fig.7 Singular values of system uncertainty and weighting functions

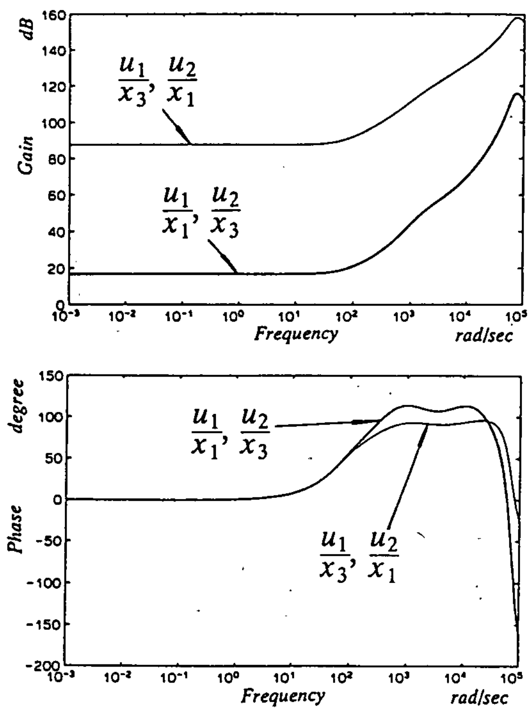


Fig.5 Singular values of H^{∞} controller of robust stability

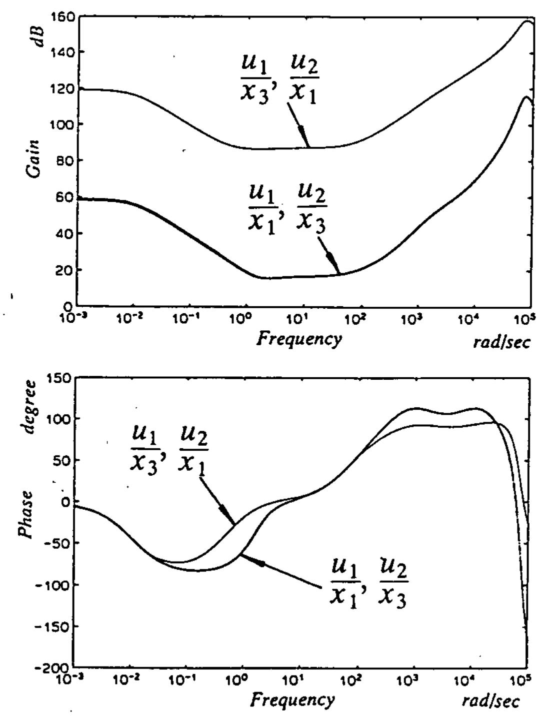


Fig.6 Singular values of H^{∞} controller of mixed sensitivity

Table 2. Poles of closed loop system

mode	without control	robust stability	mixed sensitivity
R–1	±34856 ± 0.0j	–41835 + 0.0j –41640 + 0.0j	–41835 + 0.0j –41640 + 0.0j
R–2	±35052 ± 0.0j	–34856 ± 1.4j	–34856 ± 1.4j
F–1	–0.19 ± 189.2j	–0.26 ± 189.2j	–0.25 ± 189.2j
F–2	–1.05 ± 1052.2j	–1.36 ± 1051.8j	–1.36 ± 1051.8j
F–3	–2.32 ± 2321.1j	–3.10 ± 2318.2j	–3.08 ± 2318.2j
F–4	–4.82 ± 4823.5j	–6.86 ± 4809.7j	–6.83 ± 4809.7j
current	–71.52 ± 0.0j	–85.30 ± 0.0j	–85.35 ± 0.0j

R–i: *rigid mode for ith order*, F–i: *flexible mode for ith order*

Table 3. Poles of H^{∞} controller

robust stability	mixed sensitivity
–88122 + 0j –87687 + 0j	–88129 + 0j –87694 + 0j
–21258 ± 68426j	–21366 ± 68748j
–21363 ± 68743j	–21261 ± 68432j
–3459 ± 0j	–3462 ± 0j
–1090 ± 0j	–1086 ± 0j
–155.4 ± 0j	–155.7 ± 0j
	–2.00 ± 0j –0.01 ± 0j

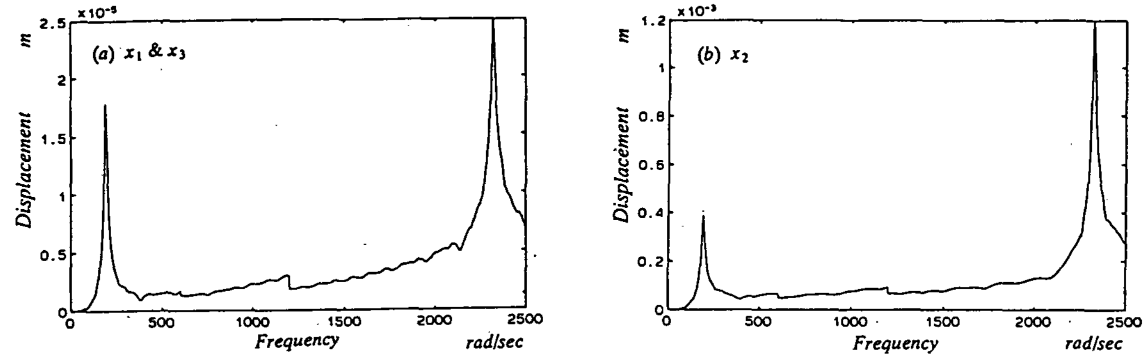


Fig.8 Frequency responses of flexible rotor to unbalanced force

Table 4. Poles of closed loop system

mode	without control	robust stability	mixed sensitivity
R–1	±34856 ± 0.0j	−34856 ± 0.14j	−34856 ± 0.14j
R–2	±35052 ± 0.0j	−35052 ± 0.14j	−35052 ± 0.14j
F–1	−0.19 ± 189.2j	−0.22 ± 189.2j	−7.14 ± 191.6j
F–2	−1.05 ± 1052.2j	−3.23 ± 1054.5j	−3.25 ± 1054.5j
F–3	−2.32 ± 2321.1j	−8.27 ± 2321.4j	−8.28 ± 2321.4j
F–4	−4.82 ± 4823.5j	−19.68 ± 4813.6j	−19.67 ± 4813.6j
current	−71.52 ± 0.0j	−130.45 + 0.0j −202.44 + 0.0j	−127.74 + 0.0j −201.11 + 0.0j

R–i: *rigid mode for ith order*; F–i: *flexible mode for ith order*

Table 5. Poles of H^∞ controller

robust stability	mixed sensitivity
−84393 + 0j −83956 + 0j	−84397 + 0j −83961 + 0j
−19811 ± 66076j	−19707 ± 65764j
−19705 ± 65761j	−19812 ± 66079j
−1810 ± 1621j	−1809 ± 1621j
−1809 ± 1621j	−1809 ± 1621j
−0.19 ± 189.2j	−94.60 ± 163.9j
	−13.48 ± 183.3j
	−13.47 ± 183.3j
	−11.53 ± 136.9j
	−11.52 ± 136.9j

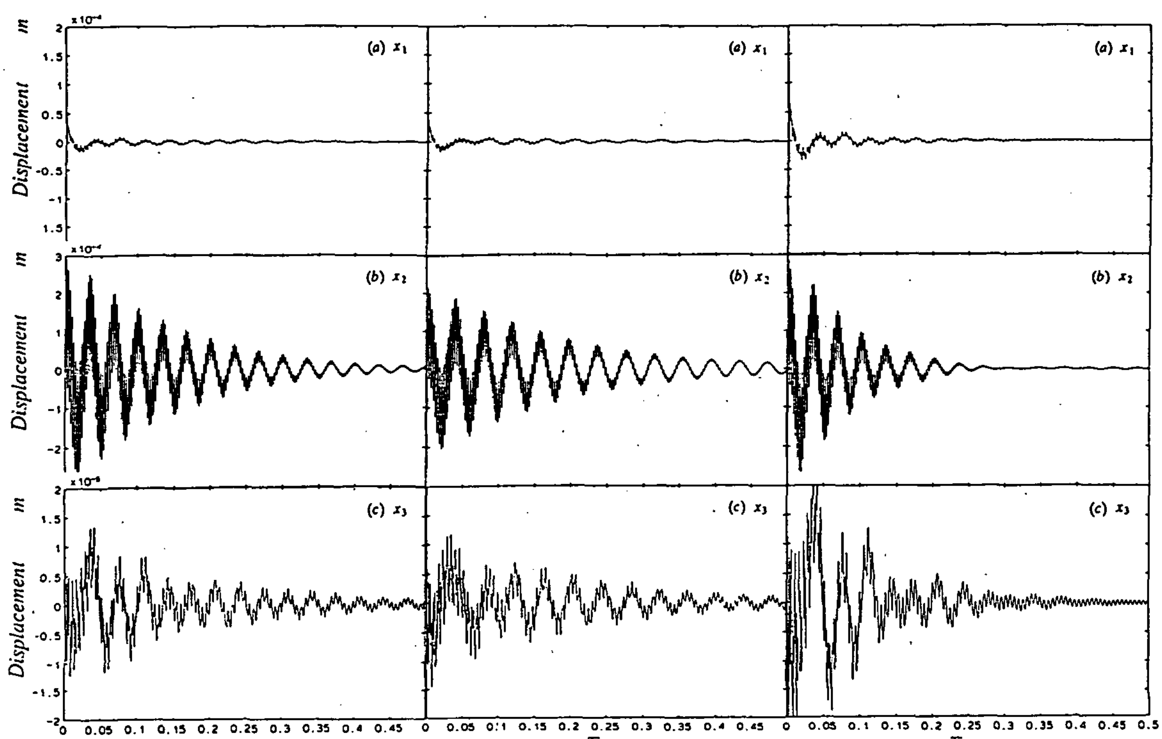


Fig.9 Time history responses at lift off

Fig.10 Time history responses at lift off (case of $m_2' = 2m_2$)

Fig.11 Time history responses at lift off (case of $p_0' = 0.5p_0$)