On Achievable H^{∞} Disturbance Attenuation in AMB Control

R. HERZOG[1] AND H. BLEULER[2]

ABSTRACT

We consider the control of AMB (active magnetic bearing) systems where we investigate the limit of achievable wideband (i.e. $\mathcal{H}^{\infty}$) disturbance attenuation. *Either* the AMB system may admit *nearly total* disturbance rejection (at the expense of high controller gain), *or* there is a *nontrivial bound* which cannot be surpassed by any stabilizing controller. The relative position of actuator and disturbance may strongly influence the achievable disturbance attenuation. The case distinction *"trivial"* $(= 0)$ versus *"nontrivial"* $(\neq 0)$ bound of achievable disturbance attenuation is related to the presence of right half-plane (RHP) transmission zeros in the plant cross transfer functions between actuator and disturbance. The main result states that these RHP transmission zeros cause a nontrivial bound of achievable $\mathcal{H}^{\infty}$ disturbance attenuation. We present some examples where this bound is computed by standard $\mathcal{H}^{\infty}$ techniques.

1 GENERAL INTRODUCTION

The design of a controller which stabilizes a given plant and meets a given set of engineering specifications can be quite a challenge [4] even in the linear time-invariant case. In particular, this turns out to be true for AMB systems with demanding performance specifications. The problem we investigate in this paper is to focus on a *single* specification regarding wide-band disturbance attenuation, and to examine *whether* or *not* this specification can be arbitrarily tightened without loosing closed-loop stability.

The industrial applications we have in mind are *electromagnetically* supported milling spindles [Sieg90]. The cutting forces of the milling process induce *wide-band* frequencies which may cause intolerable vibrations of the milling tool. These vibrations should be *attenuated* through the control of the electromagnetic bearings. The cutting forces appear as highly "unpredictable" exogenous disturbances since it is almost impossible to set up a useful and accurate modeling of the cutting process. Therefore, it is of interest to *minimize* the effect of *worst case* disturbances. This leads to the use of $\mathcal{H}^{\infty}$ theory.

[1] R. Herzog, Mechatronics Lab, ETH Zurich, Switzerland,
E-mail: raoul@ifr.ethz.ch
[2] H. Bleuler, Institute of Industrial Science, University of Tokyo, Japan,
E-mail: aa22700@kappa.iis.u-tokyo.ac.jp

2 INTRODUCTION TO $\mathcal{H}^\infty$

This section provides a motivation for $\mathcal{H}^\infty$ control and gives some related cultural remarks. An intuitively most appealing motivation for $\mathcal{H}^\infty$ control is supplied by its differential game analogy [6]: consider a plant $\boldsymbol{P}$ with two inputs, see figure 1. These two inputs which are also called players are set up in opposition. The first player $u(t)$ (that's you in fact !) is the control input generated by the controller (your strategy). The second player $w(t)$ (your adversary) is an exogenous input signal which could represent some disturbance. Of course, this exogenous input signal is a priori *unknown* to you; the only available signal for the controller input is the measured plant output $y(t)$. Your objective is to minimize the "worst case" disturbance, that is to minimize the disturbance which causes the maximal "damage" (in terms of energy) to the plant output $z(t)$. This "minimax" optimization problem characterizes $\mathcal{H}^\infty$ control.

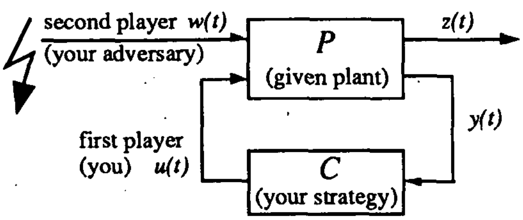

Figure 1: Game analogy for $\mathcal{H}^\infty$ control

3 PURPOSE OF THIS PAPER

We believe that most challenging AMB applications are those of *flexible* supported bodies. Furthermore, any realistic AMB design must account for the presence of *unknown* exogenous disturbances. The crucial point is that disturbances do in most cases *not* act on the same location as the AMB actuator. Consider for example the flexible shaft of an AMB milling spindle. For feasibility reasons the AMB's cannot be placed arbitrarily close to the milling tool. One might call this a *"non-collocation"* between actuator and *disturbance*. The term of non–collocation usually means that actuators and *sensors* are mounted on different locations. In this paper, the relative position of actuator and *disturbance* will play an important role.

In [8] we investigated a wide–band disturbance attenuation problem where the "supported body" was a *two-mass oscillator*, see figure 2. We were seeking for controllers $C(s)$ that

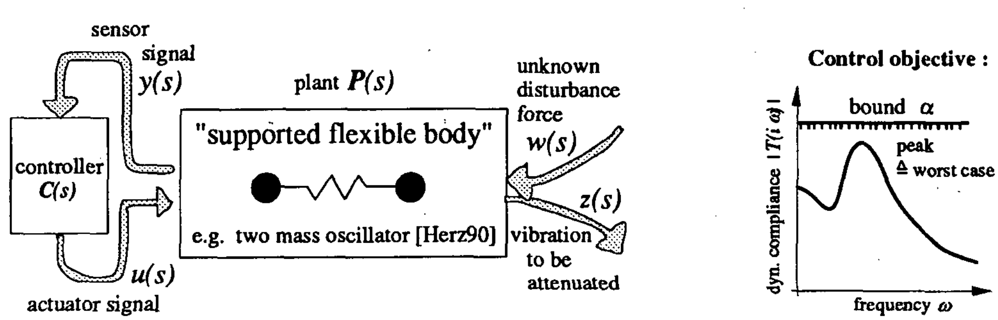

Figure 2: The objective is disturbance attenuation.

stabilize the two mass oscillator and maintain the magnitude of dynamic compliance $|T(i\,\omega)|$

uniformly below a given bound α, i.e. $|T(i\omega)| < \alpha$, for all frequencies ω. For the two–mass oscillator [8] this objective can be arbitrarily well done at the expense of appropriately high controller gain. But is this *generally* true? The purpose of this paper is to answer this question, i.e. to investigate wide–band disturbance attenuation for a more *"general"* class of flexible AMB supported bodies. In particular, this leads to the following
question: Is it *always* possible to achieve a disturbance attenuation bound α *arbitrarily* close to zero at the expense of appropriately high controller gain?
The answer will turn out to be *"yes"* or *"no"*, depending only on the plant. We attempt to characterize either case in this paper.

4 CLASSES OF MECHANICAL PLANTS

This section is devoted to the description of certain classes of mechanical plants. Let us start with a standard definition [1], [3].

Definition. A $p \times p$ matrix $\boldsymbol{H}(s)$ whose entries are *rational* functions of s with *real* coefficients is termed *positive real* if *no* entry has any pole in the open right half-plane RHP := $\{s \in \mathcal{C} : \Re e(s) > 0\}$, and if the following definiteness property holds:

$$\boldsymbol{H}(s) + \boldsymbol{H}^\star(s) \geq 0, \quad \text{for all } s \in \text{RHP}$$

Here, * denotes complex conjugate transposition. It can be shown that positive real functions arise from the modeling of certain mechanical plants with *collocated* sensor–actuator pairs. More precisely, consider the following class of mechanical transfer functions:

$$\boldsymbol{H}(s) := \boldsymbol{C}\left(s\boldsymbol{M} + \boldsymbol{D} + \boldsymbol{G} + s^{-1}\boldsymbol{K}\right)^{-1}\boldsymbol{B} \tag{1}$$

where $\boldsymbol{M}, \boldsymbol{D}, \boldsymbol{G}, \boldsymbol{K}, \boldsymbol{B}, \boldsymbol{C}$ are constant real matrices with the 5 properties:

1) mass matrix $n \times n$	$\boldsymbol{M}$	$=$	$\boldsymbol{M}^t$	$>$	0
2) damping matrix $n \times n$	$\boldsymbol{D}$	$=$	$\boldsymbol{D}^t$	$\geq$	0
3) gyroscopic matrix $n \times n$	$\boldsymbol{G}$	$=$	$-\boldsymbol{G}^t$		
4) stiffness matrix $n \times n$	$\boldsymbol{K}$	$=$	$\boldsymbol{K}^t$	$\geq$	0
5) collocated influence matrix $n \times p$	$\boldsymbol{B}$	$=$	$\boldsymbol{C}^t$		

From the "positive real lemma" [1] it follows that such transfer matrices $\boldsymbol{H}(s)$ are positive real. In particular, the positive realness of collocated mechanical systems implies the "minimum-phase" property, i.e. there are *no* transmission zeros in the RHP.

As already mentioned in section 3, we are often confronted with a *non*–collocation between actuator and *disturbance*. Therefore, the plant cross transfer functions between actuator and disturbance are generally *not* positive real, and even in some cases RHP transmission zeros may occur. There is a remarkable class of non–collocated systems with still no RHP transmission zeros, namely the class of ladder structures. Basic facts of ladder structures are compiled in appendix 1. Consider for example the mechanical system in figure 3. It consists of three masses $m_1, m_3, m_5 > 0$ moving along the horizontal direction, three springs $c_2, c_4, c_{15} \geq 0$, and one damper $d_{15} \geq 0$. Let $w(t)$ denote an input force acting on mass m_5 which causes a system motion with velocity $y(t)$ at mass m_1. Note that the components c_{15} and d_{15} "skip" mass m_3. If $c_{15} = d_{15} = 0$ the system in figure 3 has ladder structure, and the corresponding transfer

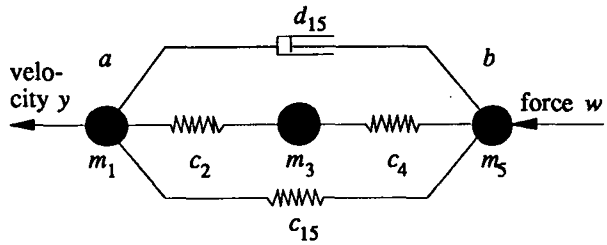

Figure 3: This is a ladder structure if $c_{15} = d_{15} = 0$.

function has certainly *no* RHP transmission zeros. If $c_{15} \neq 0$ and/or $d_{15} \neq 0$ the transfer function *may* have, but does not necessarily have, RHP transmission zeros. It depends on the precise parameter values of all masses, springs and dampers.

5 PROBLEM SETUP

Consider the feedback setup in figure 2. Plant $\boldsymbol{P}$ has two inputs w, u and two outputs z, y where w denotes an unknown disturbance, u is the actuator signal, z is the unmeasured vibration signal to be attenuated, and y is the sensor signal. The feedback interconnections are described by

$$\begin{aligned} \begin{bmatrix} z(s) \\ y(s) \end{bmatrix} &= \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} \\ u(s) &= C(s)\,y(s) \end{aligned} \tag{2}$$

The closed–loop transfer function from disturbance w to signal z is denoted by $T(s)$. In our context the mechanical meaning of $T(s)$ is "dynamic compliance". The following *linear fractional map* gives the one–to–one correspondence between controller $C(\cdot)$ and compliance $T(\cdot) = T(P_{ij}(\cdot), C(\cdot))$:

$$T = P_{11} + P_{12}C\,(1 - P_{22}C)^{-1}\,P_{21} \tag{3}$$

The *control objective* is to stabilize $\boldsymbol{P}$ and to find the limit of achievable disturbance attenuation with respect to the infinity norm. The infinity norm $\| T \|_\infty$ of $T(\cdot)$ denotes the *"worst case"* compliance defined by its peak value in the frequency response.

$$\| T(\cdot) \|_\infty \overset{\text{def}}{:=} \sup_{0 \le \omega < \infty} |T(i\omega)|$$

Recall a few notions related to stability of feedback systems. The stability of a closed–loop setup $[\boldsymbol{P}, C]$ is understood in the strict sense of internal stability [7]. The set $\mathcal{RH}^\infty$ consists of those rational transfer functions which have no poles in the *extended closed* right half–plane $\overline{\text{RHP}}_e := \{s \in C : \Re(s) \ge 0\} \cup \{\infty\}$. Finally, $\mathcal{S}(\boldsymbol{P})$ denotes the *set* of all linear controllers $C(s)$ which *stabilize* $\boldsymbol{P}(s)$.

Problem:

i.) For a given $\boldsymbol{P}$ solve the optimization problem:

$$\breve{\alpha} := \inf_{C \in \mathcal{S}(\mathcal{P})} \| T(\boldsymbol{P}(\cdot), C(\cdot)) \|_\infty \tag{4}$$

ii.) Determine *whether* or *not* $\breve{\alpha}$ can be arbitrarily close to zero. $\alpha \to 0$ corresponds to nearly total disturbance rejection. Which $\boldsymbol{P}$'s do admit nearly total disturbance rejection, and how may the mechanical realization of such $\boldsymbol{P}$'s look like?

The solution of i.) has been known for some years to the control community; its mathematics even originate from the first decade of this century (see references within [7]). A concrete example of problem i.) is solved in section 7. The next section answers problem ii.).

6 THE MAIN RESULT

The following theorem characterizes those plants $\boldsymbol{P}$ which admit nearly total disturbance rejection control without loosing closed–loop stability.

Theorem. Assume that $\boldsymbol{P}$ is *stable* and *strictly proper*, i.e. $\boldsymbol{P} \in \mathcal{RH}^\infty$ and $\boldsymbol{P}(\infty) = \boldsymbol{0}$. Furthermore assume that the cross transfer functions $P_{12}(s)$ and $P_{21}(s)$ have no finite $\overline{\text{RHP}}$ zeros, which means that $(s+1)^{-r} \cdot P_{12}^{-1}$ and $(s+1)^{-r} \cdot P_{21}^{-1} \in \mathcal{RH}^\infty$ for some positive integer r. Then, there exist a sequence of *stabilizing* controllers $C_k \in \mathcal{S}(\boldsymbol{P})$ with the following minimizing property:

$$\lim_{k \to \infty} \| T(\boldsymbol{P}(\cdot), C_k(\cdot)) \|_\infty = 0 = \breve{\alpha} \tag{5}$$

We call this *"nearly total disturbance rejection"*.

The proof of this theorem is in appendix 2. In general the unique controller $C^\star(s)$ yielding

$$T(\boldsymbol{P}(s), C^\star(s)) \equiv 0 \tag{6}$$

exists regardless of plant $\boldsymbol{P}$, see e.g. figure 4, but it is usually *not* stabilizing. The key point is that if $\boldsymbol{P}$ has the properties of the above theorem then $C^\star \notin \mathcal{S}(\boldsymbol{P})$ can be "approximated" by a sequence of $C_k \in \mathcal{S}(\boldsymbol{P})$ in the sense of equation (5). However, it should be noticed that the sequence $C_k(s)$ of controllers is *unfeasible* because it has *unbounded* gain for high frequencies, i.e. $|C_k(\infty)| \to \infty$ as $k \to \infty$. We have to deal with this basic trade–off between disturbance attenuation and control effort. Until here we only considered a *single* specification regarding disturbance attenuation. However, a feasible controller design must take into account a number of engineering specifications, e.g.

- Limited control effort. The closed–loop poles must be prevented from "migrating" too far to the left because this corresponds to very fast closed–loop dynamics requiring both unrealistically high controller gains and bandwidths. The allowed closed–loop pole region can be *restricted* to a subset of the left half–plane, e.g. a disc [8]. This can be achieved by using bilinear transform techniques.
- Sensor and actuator noise. The "size" of several closed–loop functions must not exceed some given bounds.
- Uncertainty in the plant description [11]. Robust stability and/or robust performance for a given class of "neighboured" plants is required.

Next we characterize in terms of the *mechanical realization* a special class of $\boldsymbol{P}$'s which allow nearly total disturbance rejection.

Corollary. "Most" mechanical *ladder* structures do admit nearly total disturbance rejection. Network theory [3] has had it for a long time that transfer functions of passive ladder networks are "minimum phase", hence the corresponding transfer functions have no finite RHP zeros, see appendix 2. For the time being, we disregard the special case of imaginary axis poles of $\boldsymbol{P}$ and/or imaginary axis zeros of P_{12}, P_{21}. Therefore, mechanical ladder structures basically fulfil the assumptions of the above theorem.

7 EXAMPLE

Paradoxically, the interesting examples are rather those which do *not* fit in with the previous theorem. Therefore we consider a *non-ladder* plant, see figure 3, where we choose $m_1 = 1, c_2 = \frac{181}{500}$, $m_3 = \frac{227}{135}$, $c_4 = 1$ $m_5 = 2$, $c_{15} = \frac{108}{227}$ and $d_{15} = \frac{135}{227}$. The equations of motion lead to the following plant transfer matrix $\boldsymbol{P}(s)$:

$$\boldsymbol{P}(s) = \frac{1}{n(s)} \begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} \quad \text{where}$$

$$\begin{cases} n(s) &= 4s^2(56750s^4 + 50625s^3 + 135386s^2 + 63990s + 79790) \\ z_{11}(s) &= 113500s^4 + 67500s^3 + 187022s^2 + 54675s + 68175 \\ z_{12}(s) &= z_{21}(s) \ = \ 675(s+1)(100s^2 - 20s + 101) \\ z_{22}(s) &= 5(45400s^4 + 13500s^3 + 70274s^2 + 10935s + 13635) \end{cases}$$

Note that $s \cdot \boldsymbol{P}(s)$ is a positive real matrix function. Factor s appears since we used the *displacements* of the masses m_1 and m_5 for the signals y and z. Clearly the cross transfer functions $P_{12}(s) = P_{21}(s)$ are *non-minimum* phase, since $z_{12}(s) = z_{21}(s)$ have a double complex conjugate pair of RHP zeros at $s_r := +\frac{1}{10} + i$ and $\bar{s}_r := +\frac{1}{10} - i$. Therefore, we expect a *nontrivial bound* $\breve{\alpha}$ for the disturbance attenuation problem. The aim of the sequel is the computation of this bound.

Because of the pole at $s = 0$ (rigid-body), $P_{22} \notin \mathcal{RH}^\infty$ and equation (13) is not valid for parametrizing the set of all stabilizing controllers $\mathcal{S}(\boldsymbol{P})$. Therefore, we proceed by a coprime factorization [7] of P_{22}. Factor $P_{22} = ND^{-1}$ where N and D are coprime over $\mathcal{RH}^\infty$, i.e. there exist $X, Y \in \mathcal{RH}^\infty$ with $YD - NX = 1$. Note that $C = -1$ is stabilizing, which yields a simple solution of the Bezoutian: $D = (1 + P_{22})^{-1}$, $N = P_{22}(1 + P_{22})^{-1}$, $X = -1$ and $Y = 1$. This leads to the parametrization:

$$\mathcal{S}(P) \ = \ \{\frac{X - DQ}{Y - NQ} : Q \in \mathcal{RH}^\infty\} \ = \ \{-\frac{1 + P_{22} + Q}{1 + P_{22} - P_{22}Q} : Q \in \mathcal{RH}^\infty\} \tag{7}$$

The closed-loop function T is affine in Q, i.e. $T = T_a - T_b Q$. The $\overline{\text{RHP}}_e$ zeros of T_b are $\{s_r, \bar{s}_r, \infty\}$ with the multiplicities 2, 2 and 6 respectively. Note that the associated interpolation conditions for $T(s)$ are of both *interior* $\{s_r, \bar{s}_r\}$ and *boundary* $\{\infty\}$ type. Seeking the bound $\breve{\alpha}$ of achievable $\mathcal{H}^\infty$ disturbance attenuation amounts to solving the following minimax optimization problem:

$$\breve{\alpha} \ = \ \inf_{Q \in \mathcal{RH}^\infty} \| T_a - T_b Q \|_\infty \tag{8}$$

It is well-known [7] that the solvability of the scalar-valued model matching problem (8) is ensured if $T_b(i\omega) \neq 0$ for $0 \leq \omega \leq \infty$. With our example this condition is violated at ∞. Therefore, we disregard the boundary interpolation conditions at ∞, since they do not alter

$\breve{\alpha}$. This leads to replacing $T_b(s)$ by $\widetilde{T}_b(s)$, where

$$\widetilde{T}_b(s) = \frac{(100s^2 - 20s + 101)^2}{(100s^2 + 20s + 101)^2}$$

We will now draw a brief outline of the standard "machinery" we used to compute the solution of problem (8). Readers who are not familiar with these concepts refer to [7]. At first, set up the Nehari problem $\inf_Q \| R - Q \|_\infty$ where $R = \widetilde{T}_b^{-1} T_a$. Next, decompose $R(s)$ in its stable and antistable projection: $R = R_1 + R_2$ where $R_1(s)$ is the strictly proper antistable part. Find a state space realization of $R_1(s)$:

$$R_1(s) = \boldsymbol{C}\,(s\boldsymbol{I} - \boldsymbol{A})^{-1}\,\boldsymbol{B}$$

Now compute the controllability and observability gramians $\boldsymbol{L}_c$ and $\boldsymbol{L}_o$ by solving the Lyapunov equations:

$$\boldsymbol{A}\,\boldsymbol{L}_c + \boldsymbol{L}_c\,\boldsymbol{A}^t = \boldsymbol{B}\,\boldsymbol{B}^t \quad (9)$$

$$\boldsymbol{A}^t\,\boldsymbol{L}_o + \boldsymbol{L}_o\,\boldsymbol{A} = \boldsymbol{C}^t\boldsymbol{C} \quad (10)$$

The final solution $\breve{\alpha}$ equals the square root of the largest eigenvalue of $\boldsymbol{L}_c\,\boldsymbol{L}_o$.

$$\breve{\alpha} = \sqrt{\lambda_{\max}(\boldsymbol{L}_c\,\boldsymbol{L}_o)} \approx 0.6762 \quad (11)$$

There is *no stabilizing* controller enabling a better performance than $\breve{\alpha}$, *no matter* how complex it might be (in terms of the controller order), and *no matter* how important the allowable controller gain might be. We leave the remainder of this section to analyze and interpret the "optimal" controller $\breve{C}(s)$ enabling bound $\breve{\alpha}$.

Following the computational steps described in [7] we obtainded:

$$\breve{C}(s) \approx (s^5 + 0.2613s^4 + 1.2002s^3 + 0.2315s^2 + 0.8478s + 0.3624) \,/\, (s^3 - 0.3334s^2 + 0.7375s + 0.2436)$$

Note that $\breve{C}(s)$ is *not* proper, i.e. it has *unbounded* gain at high frequencies. This happens because we disregarded the boundary interpolation conditions. Therefore $\breve{C} \notin \mathcal{S}(\boldsymbol{P})$, but there exists a sequence of *proper* and *stabilizing* controllers $C_k(s)$, i.e. $C_k \in \mathcal{S}(\boldsymbol{P})$, with the property that $\| T_k \|_\infty \rightarrow \breve{\alpha}$ as $k \rightarrow \infty$. This means that the performance may be *arbitrarily* close to bound $\breve{\alpha}$ at the expense of high controller gain.

Note again that bound $\breve{\alpha}$ is not trivial, i.e. $\breve{\alpha} \neq 0$. Disregarding stability, $\breve{\alpha} = 0$ means $T(\boldsymbol{P}(s), C^\star(s)) \equiv 0$, where $C^\star(s)$ denotes the corresponding controller. Using equation 3 results in:

$$C^\star(s) = (113500s^4 + 67500s^3 + 187022s^2 + 54675s + 68175) \,/\, (1135\,(100s^2 + 81))$$

Figure 4 helps to clarify the physical behaviour of $C^\star$. Neither $\breve{C}(s)$ nor $C^\star(s)$ are feasible controllers because they both lack any bandwidth constraint. But $\breve{C}(s)$ can be understood as a "cheap control" limit case, whereas $C^\star(s)$ is *completely* fictitious in the following sense: There is *no* sequence of *stabilizing* controllers which approaches the performance of $C^\star(s)$.

Note that $C^\star(s)$ *completely* cancels the dynamics of plant $\boldsymbol{P}(s)$. It seems that $\breve{C}(s)$ "*partially*" *cancels the* dynamics of $\boldsymbol{P}(s)$; this question is currently under investigation.

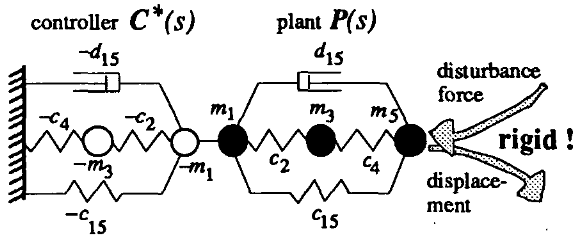

Figure 4: $C^\star$ completely cancels the dynamics of $\boldsymbol{P}$.

8 PRACTICAL ASPECTS OF $\mathcal{H}^\infty$ CONTROLLERS

The off-line effort for computing $\mathcal{H}^\infty$ controllers has been drastically reduced recently by the "state space approach" in [6], where the resulting controllers are basically given in terms of two algebraic Riccati equations. $\mathcal{H}^\infty$ software packages are already available [5], and algorithms are being improved and standardized. These advances have a common desirable consequence: the $\mathcal{H}^\infty$ approach is nowadays available to a broader section of the control community. However, there is still a computational on-line burden because standard $\mathcal{H}^\infty$ designed MIMO (multiple input multiple output) controllers are fully coupled, and their order is roughly the same as the order of the plant. That is why there remains a strong need to keep up with the latest developments in special purpose controller architectures, in modern "closed-loop" controller reduction schemes [2], and in decentralized control [12].

There is still a large gap between the development of $\mathcal{H}^\infty$ theory and practical implementations of $\mathcal{H}^\infty$ designed controllers. We believe that there are several reasons responsible for this gap. First, $\mathcal{H}^\infty$ control is a very recent topic and its mathematics is more complicated than those of LQG theory. $\mathcal{H}^\infty$ theory basically provides *"box-with-crank"* tools for solving $\mathcal{H}^\infty$ "standard problems". The crucial point which needs a big amount of control engineering knowledge is the *setup* rather than the *solution* of the $\mathcal{H}^\infty$ "standard problem", since the setup requires a profound understanding of the uncertainties in both physical plant and disturbance descriptions [11].

Experimental results based on the $\mathcal{H}^\infty$ "mixed sensitivity" approach were shown in [9]. At our Institute, experimental setups are planned to be effectuated soon. AMB milling spindles would be a particularly challenging application example for $\mathcal{H}^\infty$ control since the cutting forces of the milling process appear as a highly unpredictable exogenous input.

9 CONCLUSIONS

We considered the control objective of wideband disturbance attenuation. This work gives explicit and very general limits on the best achievable performance of AMB controllers. The limit does not depend on the controller type, it only depends on the structure of the given plant. This paper thus offers an absolute measure against which the performance of any given controller can be checked.

Depending on the plant this objective *can* or *cannot* be arbitrarily well done. We related this case distinction to certain plant properties, namely to the presence of certain RHP transmission zeros in the plant cross transfer functions between actuator and disturbance. Consequently, the relative position of actuator and disturbance may strongly influence the achievable disturbance attenuation. The results of this paper were obtained by the combination of topics from classical network theory and $\mathcal{H}^\infty$ theory.

APPENDIX 1: LADDER STRUCTURES

We borrowed the term "ladder structure" from classical network theory [3], where it stands for a network topology of the following type:

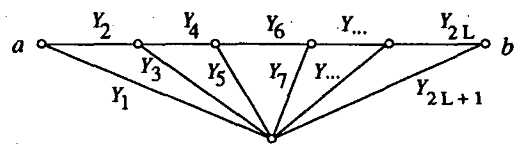

Figure 5: Graph of a ladder network.

The "branch and shunt" admittances in figure 5 are denoted by $Y_k(s)$, $k = 1, \ldots, 2L+1$. Let $H_{ab}(s)$ denote the impedance between a and b. Recall that a *tree* denotes a network path which has no loops and which passes through all nodes [3]. Now we may define a *mechanical ladder structure* in an analogous way to network theory:

Definition. A realization of a mechanical system is called *ladder* if its transfer function $H(s) = H_{ab}(s)$ can be expressed as

$$H(s) = \frac{\prod_{k=1}^{L} Y_{2k}(s)}{\sum_{\text{all trees}} \prod \text{ those } Y_k(s) \text{ forming a tree}} \tag{12}$$

where *all* the $Y_k(s)$, $k = 1, \ldots, 2L+1$ are *positive real* functions. In general, $H(s)$ itself is *not* positive real. Take for example $L = 2$. Then, equation (12) becomes $Y_2Y_4 \,/\, (Y_1Y_2Y_4 + Y_1Y_3Y_4 + Y_1Y_2Y_5 + Y_1Y_3Y_5 + Y_1Y_4Y_5 + Y_2Y_3Y_4 + Y_2Y_3Y_5 + Y_2Y_4Y_5)$. Consider for example the mechanical system in figure 3. If $c_{15} = d_{15} = 0$ the system has ladder structure and the transfer function $H(s)$ from w to y can be computed by using formula (12) with the values $Y_1(s) = m_1 \cdot s$, $Y_2(s) = \frac{c_2}{s}$, $Y_3(s) = m_3 \cdot s$, $Y_4(s) = \frac{c_4}{s}$ and $Y_5(s) = m_5 \cdot s$.

Network theory [3] has had it for a long time that (passive) ladder networks are "minimum phase", hence the corresponding transfer function $H(s)$ has *no* (finite) RHP zero. In fact, this property follows from equation (12).

APPENDIX 2: PROOF OF THE MAIN RESULT

Because of the assumption $P \in \mathcal{RH}^\infty$, the set $\mathcal{S}(P)$ of all stabilizing controllers is easily parametrized [7] by

$$\mathcal{S}(P) = \{-Q(1 - P_{22}Q)^{-1} : Q \in \mathcal{RH}^\infty\} \tag{13}$$

Insert (13) in (3) which shows that T is *affine* in Q:

$$T = P_{11} - P_{12}QP_{21} \tag{14}$$

In literature this equation is associated with the "model matching problem" [7]. Equation (14) imposes interpolation conditions on $T(s)$ in the following way: the $\overline{\text{RHP}}_e$ zeros of P_{12} and

P_{21} cannot be cancelled by poles of Q, because Q is restricted to $\mathcal{RH}^\infty$. At these specified points in the $\overline{\mathrm{RHP}}_e$, T interpolates the values of P_{11} (including its derivative values according to the multiplicities of the $\overline{\mathrm{RHP}}_e$ zeros of P_{12}, P_{21}). Now the key point of the above theorem is the *assumption* that P_{12}, P_{21} have *no* $\overline{\mathrm{RHP}}_e$ zeros except for $s = \infty$. Therefore, (14) amounts only to *"boundary interpolation at infinity"*. Now consider the following sequence of Q_k's:

$$Q_k := (k^{-1}s+1)^{-2r} P_{12}^{-1} P_{11} P_{21}^{-1} \tag{15}$$

Note that $Q_k \in \mathcal{RH}^\infty$ for $k > 0$ because of the assumptions on $P_{12}(s)$ and $P_{21}(s)$. The sequence of Q_k's generates via (13) a sequence of stabilizing controllers C_k's. Inserting (15) in (14) results in:

$$T_k(s) = (1 - (k^{-1}s+1)^{-2r}) \cdot P_{11}(s) \tag{16}$$

Now $T_k(\infty) = 0$, since we assumed that $P_{11}(\infty) = 0$. Furthermore, $|T_k(i\omega)| \to 0$ pointwise as $k \to \infty$. Consequently, $\| T_k \|_\infty \to 0$ as $k \to \infty$, which completes the proof.

REFERENCES

1. Anderson, B. D. O. and S. Vongpanitlerd, *"Network Analysis and Synthesis. A modern systems theory approach"*, Prentice Hall 1973.

2. Anderson, B. D. O. and Y. Liu, *"Controller Reduction: Concepts and Approaches"*, IEEE Trans. AC, Vol. 34, No. 8, p. 802 – 812, 1989.

3. Balabanian, N. and T. Bickart, *"Linear Network Theory: Analysis, Properties, Design and Synthesis"*, Matrix Series in Circuits and Systems, 1981.

4. Boyd, S. and C. Barrett, *"Linear Controller Design: Limits of Performance"*, Prentice Hall, Information & System Sciences Series, 1991.

5. Chiang, R. Y. and M. G. Safonov, *"User's Guide of the Robust Control MATLAB Toolbox"*, The Mathworks Inc., Natick, MA 01760, 1988.

6. Doyle, J. C., K. Glover, P. P. Khargonekar and B. A. Francis, *"State Space Solutions to Standard $\mathcal{H}^2$ and $\mathcal{H}^\infty$ Control Problems"*, IEEE Trans. Contr. Aug. 1989, Vol. 34, No. 8, p. 831–847.

7. Francis, B. A. *"A Course in $\mathcal{H}^\infty$ Control Theory"*, Lecture Notes in Control and Information Sciences, Springer 1987.

8. Herzog, R. and H. Bleuler, *"Stiff AMB Control using an $\mathcal{H}^\infty$ Approach"*, 2nd Int. Symp. on MB's, Tokyo 1990.

9. Matsumura, F., M. Fujita and M. Shimizu, *"$\mathcal{H}^\infty$ Robust Control Design for a Magnetic Suspension System"*, 2nd Int. Symp. on MB's, Tokyo 1990.

10. Siegwart, R., A. Traxler and R. Larsonneur, *"Design and Performance of a High Speed Milling Spindle in Digitally Controlled Active Magnetic Bearings"*, Proc. 2nd Int. Symp. on MB's, Tokyo 1990.

11. Skelton, R.E. *"Model Error Concepts in Control Design"*, Int. J. Control, Vol. 49, No. 5, 1989.

12. Wu, Q. *"An Application of $\mathcal{H}^\infty$ Theory to Decentralized Robust Control"*, thesis ETH Zurich, No. 9116, 1990.