Transfer Function Zeros in Noncollocated Flexible Rotor Models

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ABSTRACT

Active magnetic bearing applications in rotating equipment often require that the feedback sensor be separated axially from the magnetic actuator. The stability of the resulting closed loop system can be understood by examining the nature of the plant transfer function between actuator and sensor. In this work, it is demonstrated that when the rotor has negligible damping, such transfer functions can have zeros in the right half of the complex plane, leading to very difficult stabilization and an inherent lack of robustness. This somewhat unexpected result is interpreted in terms of physical response.

A computational approach is outlined for computing the zeros and validation is provided by comparison to the exact solution for a uniform beam. In addition, the forced response of a more complex rotor is examined, illustrating the physical significance of real zeros.

INTRODUCTION

Active magnetic bearings provide a very appealing mechanism for supporting shafts in rotating machinery. Their advantages are widely touted and include reduced parasitic power loss, elimination of lubrication systems, improved lifetime, higher stable operating speeds, and better control of rotor dynamics compared to fluid film or rolling element bearings. One of the chief distinctions between active magnetic bearings and conventional passive fluid film or rolling element bearings is the need for explicit feedback of the shaft position in order to determine the dynamic character of the bearing. In many cases, it is not possible to integrate the position sensor into the structure of the magnetic actuator and it must be displaced from the actuator along the shaft axis. This displacement, often referred to as sensor-actuator noncollocation introduces some important properties into the plant dynamics which substantially complicate feedback stabilization.

The general properties of the noncollocation problem are easiest to understand when

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NOMENCLATURE

A	beam cross sectional area	K_1	actuator spring rate	$\underline{x}$	vector of beam responses
$\underline{b}$	input vector	κ_1	spring rate, nondim.	y	sensor output
$\underline{c}^T$	output vector	K_2	free end spring rate	z	axial position along beam
E	elastic modulus	κ_2	spring rate, nondim.	α	basis function amplitudes
$\underline{e}$	unit vector	M	mass matrix	β	nondimensionalizing factor
f	forcing function	L	beam length	λ	eigenvalue
$G(s)$	plant transfer function	N	spatial solution, PDE	$\phi(t)$	nondimensional force
$H(s)$	controller transfer function	P	temporal solution, PDE	$\Phi(s)$	Laplace transform of $\phi(t)$
i	$\sqrt{-1}$	s	Laplace variable	ρ	mass density of shaft
I_{xx}	area moment of inertia	t	time		
K	stiffness matrix	w	beam deflection		

the plant (rotor) has only one control input and only one sensor output. Typically, rotors will have at least two radial bearings so that the planar problem (neglecting gyroscopic effects) has at least two inputs and two outputs. However, if one of the bearings is treated as a simple spring, then the plant model becomes single input – single output (SISO). Such a system is illustrated in Figure 1.

The stabilization problem is to find a filter, $H(s)$, which transforms the output from the sensor into an appropriate controlling signal for the actuator so that the closed loop system is stable, ie.: all of its eigenvalues lie in the left half of the complex plane. Figure 2. illustrates the relationship between the plant model, $G(s)$, and the feedback filter, $H(s)$. The closed loop system transfer function is given by

$$G_{cl}(s) = \frac{G(s)}{1 + G(s)H(s)} \tag{1}$$

It is well established [8, 4] that, if the poles of $G(s)$ all lie on the $j\omega$ axis and all of the zeros of $G(s)$ are interleaved with the poles in such a manner that exactly one zero lies between any pair of poles on either the positive or negative axis (a condition referred to as pole-zero interlacing) then the system is stabilized by *any* compensator $H(s)$ whose phase lies between 0 and 180° for all frequencies. Such interlaced systems are robustly stabilizable. On the other hand, if this condition is not satisfied then the controller dynamics must be very carefully matched to the plant dynamics in order to stabilize the system. This careful matching naturally implies a lack of robustness: small parametric changes in the plant may lead to instability of the closed loop system.

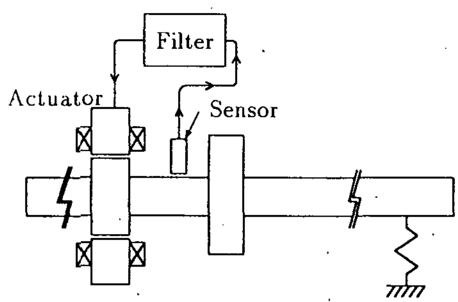

Figure 1: Rotor Schematic

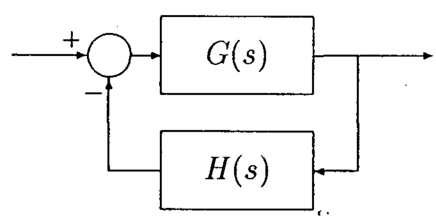

Figure 2: System Block Diagram

CONTINUOUS BEAM MODEL ZEROS

The basic properties of the transfer function from actuator to sensor can be seen by considering a uniform beam with the actuator acting at the left end and a simple spring bearing at the right end. A similar model is extensively discussed by Spector and Flashner [7] but for a moment actuator rather than a lateral force actuator and without the spring. In the following discussion, the magnetic actuator is assumed to add a destabilizing force proportional to the displacement, consistent with the usual linearized model found in the literature [2, 3, 5, 6]. Our purpose in presenting this model, which is only useful for very simple geometries, is to support the somewhat surprising conclusion that the transfer function for noncollocated systems has zeros on the positive real axis. A more useful numerical approach, based on a finite element model which is suitable for modeling complex geometries is presented in the next section.

The beam deflection, $w(z,t)$, can be modeled by the Euler-Bernoulli equation

$$\frac{\partial^2}{\partial z^2} EI_{xx} \frac{\partial^2 w}{\partial z^2} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \quad : \quad 0 < z < L \tag{2}$$

subject to the boundary conditions

$$\left.\frac{\partial^2 w}{\partial z^2}\right|_0 = 0 \quad , \quad \left.\frac{\partial}{\partial z} EI_{xx} \frac{\partial^2 w}{\partial z^2}\right|_0 = f(t) - K_1\, w|_0 \tag{3}$$

and

$$\left.\frac{\partial^2 w}{\partial z^2}\right|_L = 0 \quad , \quad \left.-\frac{\partial}{\partial z} EI_{xx} \frac{\partial^2 w}{\partial z^2}\right|_L = -K_2\, w|_L \tag{4}$$

which represent the moment and shear conditions at each end.

To nondimensionalize, define

$$y \doteq \frac{w}{L} \quad , \quad x \doteq \frac{z}{L} \quad , \quad \beta \doteq \sqrt[4]{\frac{\rho A L^4}{EI_{xx}}}$$
$$\kappa_1 \doteq \frac{L^3}{EI_{xx}} K_1 \quad , \quad \kappa_2 \doteq \frac{L^3}{EI_{xx}} K_2 \quad , \quad \phi(t) \doteq \frac{L^2}{EI_{xx}} f(t) \tag{5}$$

Further, note that the elastic modulus, E, and second area moment, I_{xx}, are constant for a uniform beam. After Laplace transformation, the resulting simplified nondimensional ordinary differential equation is

$$Y_{xxxx} + \beta^4 s^2 Y = 0 \quad : \quad 0 < x < 1 \tag{6}$$

subject to the boundary conditions

$$Y_{xx}|_0 = 0 \quad , \quad Y_{xx}|_1 = 0 \quad , \quad (\kappa_1 Y + Y_{xxx})|_0 = \Phi(s) \quad , \quad (\kappa_2 Y - Y_{xxx})|_1 = 0 \tag{7}$$

If we assume that the solution to (6) is separable, then its solutions have the form

$$Y(x,s) = N(x)P(s) = \alpha_j e^{\lambda_j x} P(s) \tag{8}$$

yielding the characteristic equation

$$\lambda^4 + \beta^4 s^2 = 0 \tag{9}$$

which has four solutions for any given value of s:

$$\lambda = \gamma\{1, i, -1, -i\} \quad : \quad \gamma \doteq \beta(1+i)\sqrt{0.5s} \tag{10}$$

Inversely,

$$s = \pm i\gamma^2/\beta^2 \tag{11}$$

The solution can then be written, in the frequency domain, as

$$Y(x,s) = \left\{\alpha_1 e^{\gamma x} + \alpha_2 e^{i\gamma x} + \alpha_3 e^{-\gamma x} + \alpha_4 e^{-i\gamma x}\right\} P(s) \tag{12}$$

Values for α_i and $P(s)$ are found in terms of $\Phi(s)$ by imposing the boundary conditions:

$$\left\{\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array}\right\}^{\mathsf{T}} = V^{-1}(s)\left\{\begin{array}{cccc} 0 & 0 & 1 & 0 \end{array}\right\}^{\mathsf{T}} \frac{\Phi(s)}{P(s)} \tag{13}$$

where

$$V(s) \doteq \begin{bmatrix} 1 & -1 & 1 & -1 \\ e^{\gamma} & -e^{i\gamma} & e^{-\gamma} & -e^{-i\gamma} \\ \gamma^3 + \kappa_1 & -i\gamma^3 + \kappa_1 & -\gamma^3 + \kappa_1 & i\gamma^3 + \kappa_1 \\ (\gamma^3 - \kappa_2)e^{\gamma} & (-i\gamma^3 - \kappa_2)e^{i\gamma} & (-\gamma^3 - \kappa_2)e^{-\gamma} & (i\gamma^3 - \kappa_2)e^{-i\gamma} \end{bmatrix} \tag{14}$$

The transfer function $Y(x,s)/\Phi(s)$ can be found, using Cramer's rule, as:

$$\frac{Y(x,s)}{\Phi(s)} = \frac{\det[W(x,s)]}{\det[V(s)]} \tag{15}$$

where

$$W(x,s) \doteq \begin{bmatrix} 1 & -1 & 1 & -1 \\ e^{\gamma} & -e^{i\gamma} & e^{-\gamma} & -e^{-i\gamma} \\ e^{\gamma x} & e^{i\gamma x} & e^{-\gamma x} & e^{-i\gamma x} \\ (\gamma^3 - \kappa_2)e^{\gamma} & (-i\gamma^3 - \kappa_2)e^{i\gamma} & (-\gamma^3 - \kappa_2)e^{-\gamma} & (i\gamma^3 - \kappa_2)e^{-i\gamma} \end{bmatrix} \tag{16}$$

The two matrices, $V(s)$ and $W(x,s)$ are the same except that the third row of $V(s)$ (where the shear force enters) is replaced by the set of basis functions $e^{\lambda_i x}$.

The zeros of the transfer function occur at those values of s for which the numerator of (15) is zero. Note that the zeros (solutions of $\det[W(x,s)] = 0$) depend upon the sensor location, x, while the poles (solutions of $\det[V(s)] = 0$) do not. Also, note that the boundary condition at the left end (represented by κ_1) has no effect on the zeros of the system. Thus, the destabilizing effect of the magnetic actuator, while altering the poles of the open loop plant, does not alter the zeros.

If we ignore the spring at the right end ($\kappa_2 = 0$), the zeros at each end of the beam can be readily found from

$$\det W(s,0) = 0 \qquad \det W(s,1) = 0$$

At the left end, it is convenient to express γ as

$$\gamma = 0.5(2a + b) - 0.5(b)i$$

where a and b are real. With this substitution, the zeros are the simultaneous solutions to

$$\tanh(a + b) = \tan a \quad \text{and} \quad \tanh a = \tan(a + b) \tag{17}$$

In the former relationship, $da/db > 0$, while for the latter, $da/db < 0$. Further, the largest upward displacement of the solutions to the former is $\pi/4$ while the latter move downward at most $\pi/4$. Thus, the largest relative displacement of the solutions is $\pi/2$. Since the solutions are all nearly π apart (and always more than $\pi/2$ apart), there is no b for which the solutions "skip index" and realign. Therefore, simultaneous solutions exist only for $b = 0$:

$$\tan \gamma = \tanh \gamma \tag{18}$$

The first root is $\gamma = 0$ and the remaining roots are slightly less than $\pi/4 + n\pi$ where n is a positive integer. Thus, at the left end the zeros are

$$s = 0, \quad \pm i\gamma^2/\beta^2 \approx 0, \pm i(\pi/4 + n\pi)^2/\beta^2 \quad n > 0 \tag{19}$$

The zeros of the transfer function at the left end all lie on the imaginary axis.

In a similar manner, to find the zeros at the right end, it is convenient to express γ as

$$\gamma = a + (a + b)i$$

where, again, a and b are real. With this substitution, the zeros at the right end are also given by the simultaneous solutions to (17). As before, this system of equations has solutions only if $b = 0$. In this manner, the zeros at the right end are

$$s = 0, \pm i(1 + i)^2 a^2/\beta^2 \approx 0, \mp 2(\pi/4 + n\pi)^2/\beta^2 \quad n > 0 \tag{20}$$

These zeros all lie on the real axis. It is interesting to note that the magnitudes of the right end zeros are precisely twice the magnitudes of the left end zeros.

The zeros of the transfer function for values of x lying between zero and one do not yield quite as readily to simple algebra, so a general solution is not presented here. In any case, they can be obtained numerically from $\det V(x, s) = 0$. The important point to note is that the zeros always lie either on the real axis or on the imaginary axis and that, if $0 < x < 1$, zeros can be found on *both* axes. Further, these properties are not modified by the presence of a positive spring, κ_2, at the right end.

FINITE ELEMENT BEAM MODEL ZEROS

Continuous beam models based on partial differential equations provide useful insight to the fundamental mathematics of the transfer function zeros problem, but are not readily applied to real engineering analysis. Instead, the usual approach is to develop a finite element model of the rotor and study the properties of the model matrices. Such a model is based on the same partial differential equation as in the preceding section, but the equation is essentially *solved* for each short axial section of the rotor subject to continuity conditions at each section interface. The advantage of such an approach is that it readily permits more complex geometries and the resulting models can easily be interfaced with more complex boundary conditions.

The finite element form of the undamped rotor (beam) model has the general form

$$M\ddot{\underline{x}}(t) + K\underline{x}(t) = b\underline{e}_a f(t) \qquad y(t) = c\underline{e}_s \underline{x}(t) \tag{21}$$

The vector $\underline{x}$ is the collection of displacements and slopes at the interface between each beam section. These interfaces are commonly referred to as mass stations since the mass of the beam is generally treated as acting as particles at these points. The actuator force, f, enters the system through the $\underline{e}_a$ vector and the sensor, y, measures the response through the $\underline{e}_s$ vector. These vectors are unit vectors with the element corresponding to the sensor or actuator location set equal to 1.0. The plant transfer function from the actuator to the sensor is found by taking the Laplace transform of these equations and solving for $Y(t)$ in terms of $F(t)$:

$$G(s) = \frac{Y(s)}{F(s)} = bc\underline{e}_s^{\mathsf{T}} \left[s^2 M + K\right]^{-1} \underline{e}_a \tag{22}$$

Using Cramer's rule to form the inverse yields

$$G(s) = bc\frac{\det\left[s^2 M + K\right]_{[s,a]}}{\det\left[s^2 M + K\right]} \tag{23}$$

The denominator of this transfer function is independent of the sensor and actuator locations and forms the characteristic polynomial for the plant; its roots are the eigenvalues of the beam, or poles of the transfer function and they characterize the homogeneous or unforced solution to the differential equation. If M and K are symmetric and positive semidefinite, then the poles of $G(s)$ all lie on the imaginary axis, indicating that the natural motions of the beam are sinusoidal without decay. A significant exception to this occurs when K is symmetric but not positive semidefinite, as is the case when the model includes a magnetic actuator (which has a negative open-loop stiffness.) In this case, two of the eigenvalues lie on the real axis, disposed symmetrically about the origin.

The numerator of the transfer function, formed by finding the cofactor of the (s, a) element (that is, strike the a^{th} row and s^{th} column) of the indicated matrix and compute the determinant of the resulting smaller matrix) depends explicitly upon the choice of s and a: the sensor and actuator locations. The roots of the numerator are the zeros of the transfer function and they characterize the particular or forced solution to the differential equation. These zeros are the solutions, λ_i of the generalized eigenvalue problem

$$\lambda_i^2 \left[M\right]_{[s,a]} \underline{\phi}_i = \left[K\right]_{[s,a]} \underline{\phi}_i \tag{24}$$

It is important to note that, if $[M]_{[s,a]}$ is singular (as it is for $s \neq a$), then the number of eigenvalues is not necessarily equal to the dimension of the matrix.

As illustrated with the continuous beam model, these eigenvalues (zeros of the transfer function) can be either real or imaginary. When the sensor and actuator are collocated, it is readily shown that all of the zeros are imaginary. [1] However, if the sensor and actuator are not collocated, then some of the zeros may be real.

At first pass, it seems unlikely that such a system would have real zeros: they are expected to be associated with dissipation and yet the model has no dissipation mechanisms. However, the physical significance of real zeros can be understood by examining the general meaning of zeros. If the transfer function $G(s)$ is written as a ratio of polynomials:

$$G(s) = \frac{Y(s)}{F(s)} = \frac{\sum \alpha_i s^i}{\sum \beta_i s^i} \tag{25}$$

then the ratio can be expanded and the inverse Laplace transform taken (ignoring initial conditions) to yield

$$\sum \beta_i y^{(i)} = \sum \alpha_i f^{(i)} \tag{26}$$

Assuming that f takes the form $Fe^{\lambda t}$, we have

$$\sum \beta_i y^{(i)} = Fe^{\lambda t} \sum \alpha_i \lambda^i \tag{27}$$

If λ is a zero of $G(s)$, then by definition

$$\sum \alpha_i \lambda^i = 0 \quad \Rightarrow \quad \sum \beta_i y^{(i)} = 0 \tag{28}$$

If y is the sum of the homogenous and particular solutions:

$$y = y_h + y_p \quad \sum \beta_i y_h^i \doteq 0 \quad y_p = Y_p e^{\lambda t} \tag{29}$$

then

$$Y_p e^{\lambda t} \sum \beta_i \lambda^i = 0 \tag{30}$$

Assuming that none of the poles of $G(s)$ coincide with the zeros of $G(s)$, the *particular* solution $Y_p = 0$ is zero. That is, y behaves as if it were unforced when the force $e^{\lambda t}$ is applied. Since y is just the displacement of the beam at a specific point, it is not necessary that the particular solution for the entire beam be zero; only the solution at a specific point (where y is measured) must be zero. The particular solution to (21) for $f = e^{\lambda t}$ is given by:

$$\underline{x} = \left[\lambda^2 M + K\right]^{-1} \underline{e}_a \tag{31}$$

If any element of this vector is zero, then the transfer function from the forced point (selected by $\underline{e}_a$) to the output point corresponding to the zero element has a zero at λ.

[1] This was demonstrated in the previous section using the continuous model. However, it is a general property of the M and K matrices, which are symmetric. When the same row and column are removed (sensor and actuator are collocated) the symmetry is retained. Further, the single element which might produce indefiniteness in the K matrix lies at the intersection of the removed row and column, so the reduced stiffness matrix is positive semidefinite, as can be shown easily from the definition of positive semidefinite matrices. This is sufficient to show that the zeros must all lie on the imaginary axis.

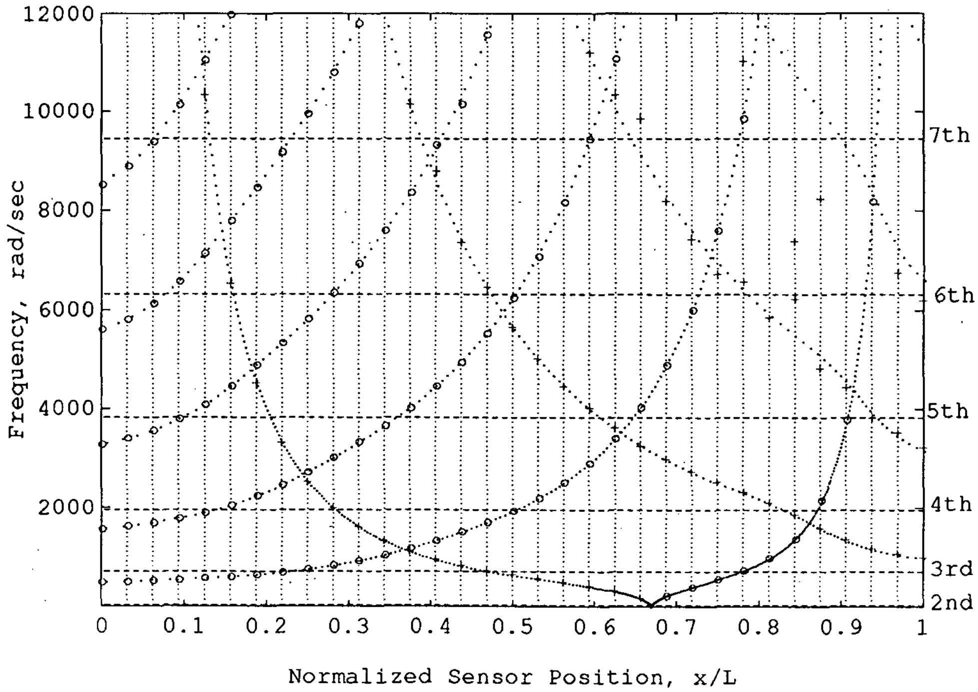

Figure 3: Poles and zeros for a free–free uniform beam

EXAMPLES

The first example serves to compare the continuous model to the finite element model. The model is a simple cylindrical beam 102 cm long and 2.5 cm in diameter with elastic modulus equal to 2.0×10^{11} N/m^2. The material mass density is taken as 7850 kG/m^3. The magnetic actuator acts at the left end and the sensor can be moved to any position along the shaft in the continuous model and in 3.3 cm increments in the finite element model.

Figure 3 shows the zeros computed both using both models when the right end of the beam is unconstrained. The circles mark the imaginary zeros of the finite element model, the crosses mark the real zeros, and the dotted curves mark the corresponding quantities for the continuous model. The poles of the finite element model are shown by the horizontal lines. The comparison is quite favorable, both for the real axis zeros and for the imaginary axis zeros. Note that, as predicted, the magnitudes of the real axis zeros with the sensor at the right end are in fact twice those of the imaginary zeros with the sensor at the left end. In addition, notice the cusp at $x/L = 0.667$ where the trajectories of a real and an imaginary zero intersect at zero. This point corresponds to the center of rotation of the free-free beam when a force is applied to the left end.

Figure 4 shows the same result when the right end of the beam is constrained by a spring (8.8×10^6 N/m). Again, the comparison is quite favorable except for the real zeros with large sensor – actuator separation. Apparently, the numerical conditioning of the

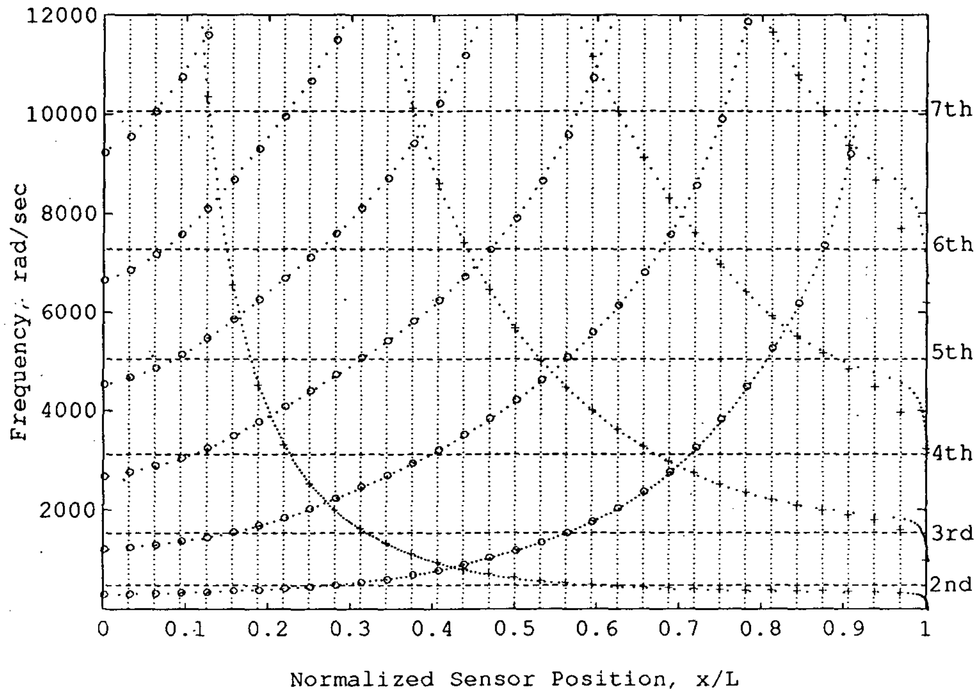

Figure 4: Poles and zeros for a uniform beam with a spring at $x = L$: $K = 8.8E6$ N/m.

model was poor for the large separation cases, leading to numerical error in the eigenvalue computation.

The second example illustrates a rotor with a more complex geometry. Here, the physical significance of real zeros examined. The rotor is a 0.64 cm diameter steel shaft 20 cm long with three disks mounted to it. The disks are mounted 3.8 cm, 8.9 cm, and 16.5 cm from the left end and have inertial properties of $m_1 = 0.054$, kG $I_{xx,1} = 5.34$E-06 kG-m^2; $m_2 = 0.20$ kG, $I_{xx,2} = 3.52$E-05 kG-m^2; and $m_3 = 0.045$ kG, $I_{xx,3} = 2.67$E-06 kG-m^2 respectively.

The rotor model has 33 mass stations: each intervening shaft segment is 0.63 cm long. A sensor is placed at the 13^{th} mass station, 7.5 cm from the left end and a force is applied at the left end. The resulting transfer function from actuator to sensor has a zero at ± 3855.38/sec. The associated forced response is shown in Figure 5. The response is largest at the point where the force is applied and goes to zero at $z = 7.5$cm and $z = 18.5$ cm.

DISCUSSION

Probably the most important point of this work is that the transfer function from an actuator to a sensor on an undamped flexible beam *always* has zeros on the real axis (and, hence, in the right half plane) when the sensor and actuator are not collocated. Clearly, such zeros substantially complicate the problem of stabilizing the system with feedback. As the separation distance increases, these zeros make their way down the real axes toward the origin, increasing their influence on closed loop stability.

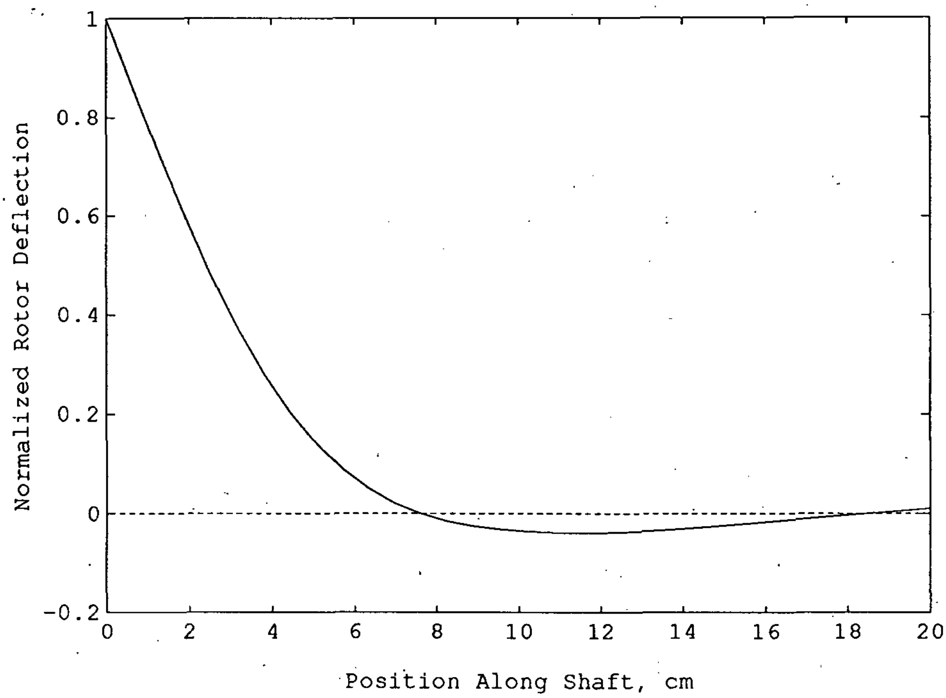

Figure 5: Forced Response of 33-element Rotor Model, $s = 3855.38$/sec

The implication is that, if rotors are truly undamped then they can not be stabilized by simple minded control algorithms such as proportional-derivative (PD) or proportional-integral-derivative (PID) control. However, if the distance between sensor and actuator is small and the rotor has dissipation mechanisms moving its poles into the left half plane, then the effect of the right half plane zeros is negligible. This result is well known to experimenters whose test rotors work fine despite the instability of their linear models.

In principle, it is possible to design a stabilizing compensator for the ugliest of systems with zeros in the right half plane. For instance, one could use sophisticated approaches such as state feedback with a Kalman filter to generate a full order controller. Such a controller would have *at least* as many poles as the plant model which could easily be forty or fifty. The drawback to this approach, setting aside the large order of the resulting controller, is that the controller will be obligated to deal strongly with the right half plane zeros. If the actual location of these zeros is not well known (as it won't be) then there is no guarantee that the resulting controller will stabilize the plant. That is, the system will not be robust to plant variation. Further, no matter how large a model is used, there are always more unmodeled zeros on the real axis. Consequently, design approaches which are inherently sensitive to these right half plane zeros are unlikely to produce satisfactory controllers.

How then, are engineers to determine whether their system will be stable when the sensor and actuator are not collocated? The answer seems to lie in better modeling of the rotor. One approach is to obtain reliable estimates of the beam dissipation, perhaps through modal testing or perhaps through improved material and assembly fit models. Another approach is to explore nonlinear beam models with an eye to determining an acceptable degree of linear instability which yields limit cycles of acceptable size.

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REFERENCES

[1] Cannon, R. H., and D. E. Rosenthal, September 1984, "Experiments in Control of Flexible Structures with Noncollocated Sensors and Actuators," AIAA Journal of Guidance and Control, VOL. 7, pp. 546-553.

[2] Hebbale, K. V., January 1985,"A Theoretical Model for the Study of Nonlinear Dynamics of Magnetic Bearings," PhD. Dissertation, Cornell University, Ithaca, NY.

[3] Humphris, R. R., Kelm, R. D., Lewis, D. W., and Allaire, P. E., October 1986, "Effect of Control Algorithms on Magnetic Journal Bearings," Trans. ASME Journal of Engineering for Gas Turbines and Power, VOL. 108, pp. 624-632.

[4] Lefante, Vincenzo, January 1992, "Noncollocation in Magnetic Bearings for Flexible Rotors", M.S. Thesis, University of Virginia, Charlottesville, VA.

[5] Maslen, E. H., and Bielk, J. R., April 1992, "A Stability Model for Flexible Rotors with Magnetic Bearings," Trans. ASME Journal of Dynamic Systems, Measurement, and Control.

[6] Schweitzer, G. and Salm, J., September 1984, "Modeling and Control of a Flexible Rotor with Magnetic Bearings," Proceedings of the Conference on Vibration of Rotating Machinery, IMECHE, York University, England.

[7] Spector, V. A. and Flashner, H., June 1990, "Modeling and Design Implications of Noncollocated Control in Flexible Systems," Transactions of the ASME, Journal of Dynamic Systems, Measurement, and Control, VOL. 112, pp. 186-193.

[8] von Flotow, A. H., and Vos, D. W., April 1991, "The Need for Passive Damping in Feedback Controlled Flexible Structures", VPI+SU Conference on Recent Advances in Active Control of Sound and Vibration.