# Controllability and Stabilizability of Unstable Objects

V. VORONKOV

## ABSTRACT.

The application of unstable objects in engineering has increased interest to the problems of stabilizing control. Magnetic bearings are one of examples of such objects since the Earnshaw-Braunbeck theorem prohibits a stable suspension of ferromagnetic body in static magnetic and gravity fields.

Restriction in control resources for an unstable object gives rise to restriction of attracting region of stabilizable equilibrium in the phase space for a system with feedback.This specificity of stabilizing unstable objects is reflected in the choice of a control-optimality criterion needing for its application the maximal attracting region at the given control restrictions.

It is shown that to use this optimizing criterion it is sufficient to obtain the object's controlability in the partial variables that are unstable. This requirement is called stabilizability.The optimal control satisfying the criterion for maximal attracting region is synthesized for the objects having a given number of different positive roots of the characteristic equation coinciding with dimensionality the vector of elementary piece-linear admissable controls has.There has been studied the structure of phase space of an optimal system; on the basis of this study there has been made some conclusion pertaining to geometry of the unstable object cortrolability region.

The results obtained have been applied to the synthesis of systems for stabilizing an elementary magnetic suspension, a magnetic suspension with elastic elements of design and with contour of eddy currents, and to stabilizing the shaft's magnetic bearings as well.

## PROBLEM STATEMENT AND INITIAL ASSUMTIONS

In well known papers [I-3] dealing with synthesizing control laws for unstable objects,alongside with commonly-accepted integral criteria there has been introduced an optimal-in-stability criterion, i.e. the criterion of reaching a maximal region

Victor Voronkov, Research Institute for Applied Mathematics and Cybernetic of Nizhni Novgorod State University after N.I.Lobatchevsky, 10 Ulyanova Str.,Nizhni Novgorod,603005,USSR tel.(8312)399532 telex 2244846 UNIGO SU

$$V\ [u^*(x)] = \max_{u\ \varepsilon\ \Omega} V\ [u(x)] = V^* \tag{1}$$

of attracting the stabllizable equilibrium x=0 in the phase state of system X at admissable controls u belonging to some restricted region $\Omega$.

In synthesizing control algorithms, the application of commonly-accepted integral optimality criteria assumes a complete controlability of the object. The aim of this paper is defining an unstable object the requirements in controlability and the conditions enabling to analitically synthesize the control being optimal in criterion (1), and also studying the structure of a phase space and that of a controlability region depending on the number of positive roots of the object's characteristic equation will have. There are considered the linearized objects

$$\dot{x}=Ax+Bu, \tag{2}$$

where

x, the n-dimensional vector of phase variables;

u, the m-dimensional vector of admissable controls; and

A,B, constant matrices of dimensions nxn, nxm respectively. The matrix A is assumed to have s different positive eigenvalues and the rest n-s values possessing negative real parts. As it is known [4], the complete object's controlability (2) is defined through the algebraic criterion

$$rank\ U = n, \tag{3}$$

where

$U = ||B\ AB\ A^2B \dots A^{(n-1)}B||\ (n \times nm)$ is a controlability matrix.

Alongside with this criterion, for an unstable object (2) there can be introduced a stabilizability criterion. For this purpose, through a nonsingular transformation of variables

$$y = Dx \tag{4}$$

the equation (2) is reduced to its canonic form

$$\begin{aligned} a)\ \dot{y}^+ &= \Lambda^+ y^+ + \nu^+ u \\ b)\ \dot{y}^- &= \Lambda^- y^- + \nu^- u \end{aligned} \tag{5}$$

where

$D\ (n \times n)$, a constant matrix;

$y^-, y^+$, stable and unstable variables of the object;

$\Lambda^+\ (s \times s), \Lambda^-\ ((n-s),(n-s))$, parts of Jordanian form of the matrix A having, respectivily, s positive and n-s eigenvalues with negative real parts; and

$\nu^+(s \times m), \nu^-((n-s), m)$, constant matrices.

The stabilizability criterion of the object (2) is defined in the form of some algebraic criterion for controlability of the unstable part (5a) of the object

$$rank\ S = s\ , \tag{6}$$

where

$S = \| \nu^{+} \Lambda^{+} \nu^{+} \Lambda^{+2} \nu^{+} \ldots \Lambda^{+(s-1)} \nu^{+} \|$, the matrix of stabilizability. This criterion, in contrast to criterion (3) of complet controlability, requires the object's controlability only in the partial variables and this is similar to the approach suggested in [ 5 ] for studing stability. Estimating the criterion (6) is a necessary condition to gain a principle capability of stabilizing an unstable object and a sufficient condition for using the technique of synthesizing an optimal-in-stability algorithm (1). So later on the criterion (6) is assumed to be valid.It is also assumed that dimention m for the admissable controls coincides with dimension s for the unstable part (5a) of the object i.e.

$$m = s \tag{7}$$

and components of this vector are simplest piece-linear functions

$$u(y) = \begin{cases} u^{+}, & \beta y \geq u^{+}, \\ \beta y, & -u^{-} \leq \beta y \leq u^{+}, \\ -u^{-}, & -u^{-} \geq \beta y, \end{cases} \tag{8}$$

where

$u^{\pm}$ , the limiting levels similar to all components of vector u;

$\beta$ $(s \times n)$, constant matrix of control coefficients; estimating them is the aim of synthesis.

## SYNTHESIS AND STRUCTURE OF THE CONTROLABILITY REGION

The assumtions assigned make it possible to analytically synthesize the control algorithm optimal in criterion (1) and through qualitative methods to study the structure of an unstable object's controlability region depending on the number of positive roots the characteristic equation of the object contains.

Analytical synthesis of the optimal algorithm is performed ( according to [1] ) from the conditions under which the entire phase space $Y^{-}$ of the stable subsystem is the region of attracting a stabilizable equlibrium. These conditions are:

- dependency of optimal control only upon unstable variables

$$u^{*} = u(\ y^{+}\ ), \tag{9}$$

- and providing asymptotic stability "in small" (local stability) of an unstable subsystem, this is obtained if matrix $\Lambda^{+} + \nu^{+}\beta$ contains eigenvalues $\hat{\lambda}$ with negative real parts

$$\mathrm{Re}\, \hat{\lambda}_j < 0 \qquad (\ j = 1,\ldots,s\ ) \tag{10}$$

The system stability "in large" (global stability) is determined through attracting region of a stabilizable equlibrium, the region (when conditions (9),(10) fulfilled) being maximal (1) and coincident with the object controlability region (2). To estimate admissable deviations of initial conditions it is sufficient in this case to study the structure of phase space $Y^+$. This structure is study depending on the number s of positive eigenvalues in matrix A being assumed different.

At the single positive eigenvalue $\lambda_1 > 0$ the unstable subsystem (5a) will be of the form

$$\dot{y}^+_1 = \lambda_1 y^+_1 + \nu^+_1 u \tag{11}$$

By choosing a sign and a scale of the controlable action, the parameter value $\nu^+_1 = -1$ is taken for convenience of the study. Since $s = 1$, then u in this case will be a scalar that, according to (9) has to be dependent only upon the unstable variable

$$u^*(y^+_1) = \begin{cases} u^+, & \beta y^+_1 \geq u^+, \\ \beta y^+_1, & -u^- \leq \beta y^+_1 \leq u^+, \\ -u^-, & -u^- \geq \beta y^+_1, \end{cases} \tag{12}$$

The characteristic equation of the linearized system (11),(12) $\chi(\hat{\lambda}) = \hat{\lambda} + \beta - \lambda_1 = 0$ has the single root $\hat{\lambda} = -\beta + \lambda_1$ and the condition (10) of asymptotic stability is valid if $\beta > \lambda_1 > 0$.The structure of phase space $Y^+$ and that of attracting region of stabilizable equlibrium $y^+_1 = 0$ are well known and shown in Fig.1. In this case the region $V^+$ will be a segment of the streight line restricted by unstable equlibrium

$$1.1 \quad y^+_1 = u^+/\lambda_1 \qquad\qquad 1.2 \quad y^+_1 = -u^-/\lambda_1$$

that correspond to control limits,which are further assumd symmetrical

$$u^+ = u^- \tag{13}$$

Unstable equlibrium states are indicated by two digits, the first denoting the quantity of control vector components subjected to restriction and the second the ordinal number of

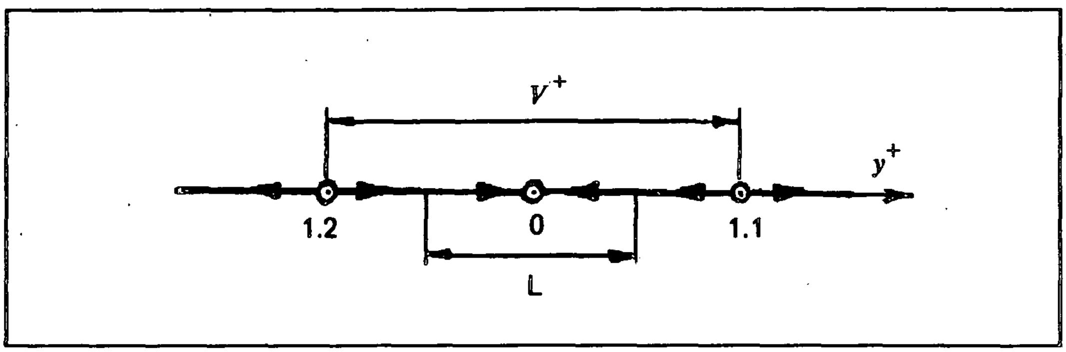



Figure 1.

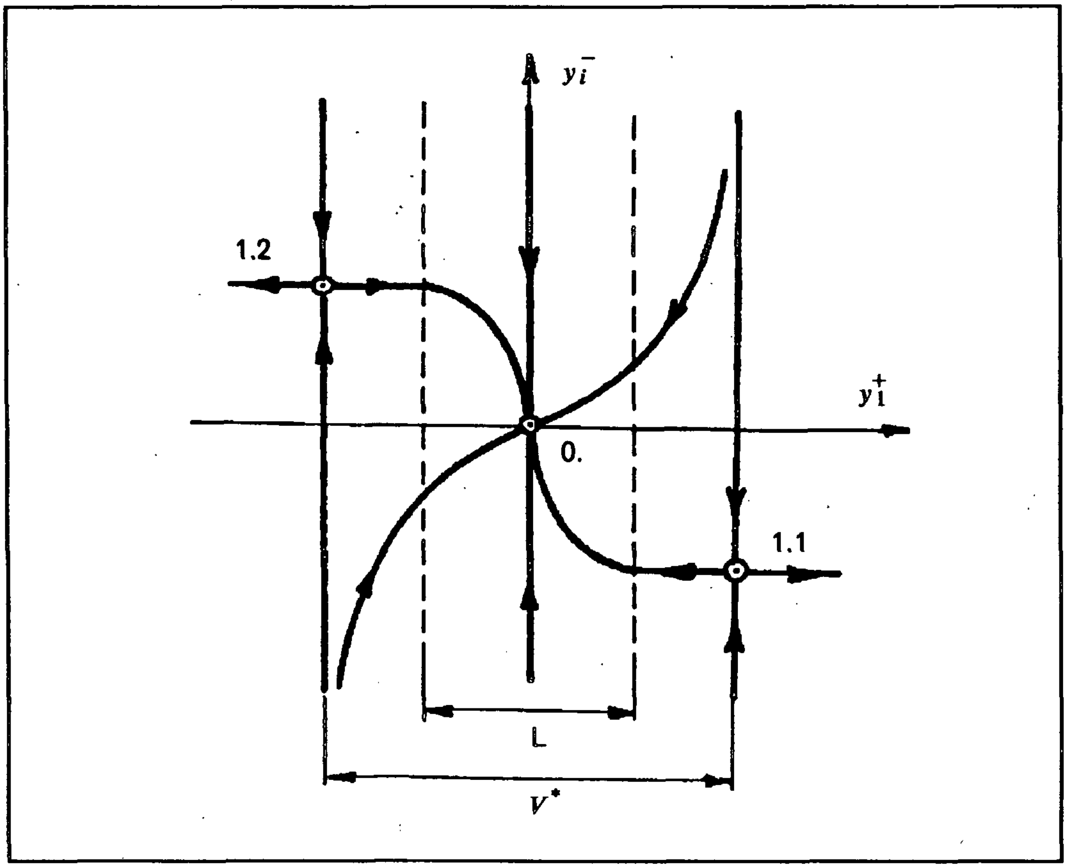



Figure 2.

equlibrium of the given type. For the subsystem (11),(12) the type of equlibrium 1.Ni will be unstable node [6] and a saddle type whith one-dimension separatrix manifold for the complete system (4); cutting the phase space Y of this system by the plane $y^+_1$ , $y^-_1$ ( where $y^-_1$ is any of the stable variables ) is shown at Fig.2. In the phase space there may be separated the region L of the system's linearity that will correspond to control linearity (12)

$$L: \qquad -u^-/\beta \leq y^+_1 \leq u^+/\beta$$

whose sizes along the axis $y^+_1$ will go down as the $\beta$ will go up it shrinks towards the equliibrium O belonging to this region. As it come from Fig.2, the boundaries of L-linearity region under optimal-in-stability control will lie parallel to separatrix manifolds of unstable equlilbriums being the boundaries of system's controlability region $V^*$.

In case of two positive eigenvalues $\lambda_1 > \lambda_2 > 0$ of matrix A, the unstable subsystem (5a) will obtain the form

$$\dot{y}_1^+ = \lambda_1 y_1^+ + \nu_{11}^+ u_1 + \nu_{12}^+ u_2$$
$$\dot{y}_2^+ = \lambda_2 y_2^+ + \nu_{21}^+ u_1 + \nu_{22}^+ u_2 \quad (14)$$

In the same way as above, through choosing signs and scales of controls the matrix $\nu^+$ is reduced to the form suitable for the study

$$\nu^+ = \begin{Vmatrix} -1 & -1 \\ -1 & 1 \end{Vmatrix}$$

To meet the condition (9) of optimal-in-stability control, the control has to be made dependent only upon unstable variables $y_1^+$, $y_2^+$, i.e. the matrix $\beta\,(2\times n)$ of parameters of admissable controls (6) has to have all columns equal to zero except the first two

$$\beta = \begin{Vmatrix} \beta_{11} & \beta_{12} & 0 & \dots & 0 \\ \beta_{21} & -\beta_{22} & 0 & \dots & 0 \end{Vmatrix} \quad (15)$$

For convenience of studying stability, signs of coefficients $\beta_{kj}$ are chosen with respect of roots of characteristic equation $\chi(\hat{\lambda}) = \hat{\lambda}^2 + a_1\hat{\lambda} + a_0 = 0$ of the system (14),(15). The roots have negative real parts when the inequalities are valid:

$$a_1 = \beta_{11} + \beta_{12} + \beta_{21} + \beta_{22} - \lambda_1 - \lambda_2 > 0$$

$$a_0 = \lambda_1\lambda_2 + 2(\beta_{11}\beta_{22} + \beta_{12}\beta_{21}) - \lambda_1(\beta_{12} + \beta_{22}) - \lambda_2(\beta_{11} + \beta_{21}) > 0$$

Under the conditions

$$\beta_{11} = \beta_0\lambda_1, \quad \beta_{12} = \beta_0\lambda_2, \quad \beta_{21} = \beta_0\lambda_1, \quad \beta_{22} = \beta_0\lambda_2, \quad (16)$$

bringing symmetry to the control algorithm, these inequalitiles will shrink to a single one

$$\beta_0 > ½ \quad (17)$$

To determine the structure of phase space $Y^+$ and that of region $V^+$ of attracting the stabilizable equlibrium

$$y_1^+ = y_2^+ = 0, \qquad u_1 = u_2 = 0$$

in the plane $y_1^+$, $y_2^+$ ( Fig. 3 ) in this case there will be coordinates of unstable equilibrium of two types:

1. $N_1$ - the first type corresponding to restricting one of the components, and
2. $N_2$ - the second type, to restricting both components of the control vector.

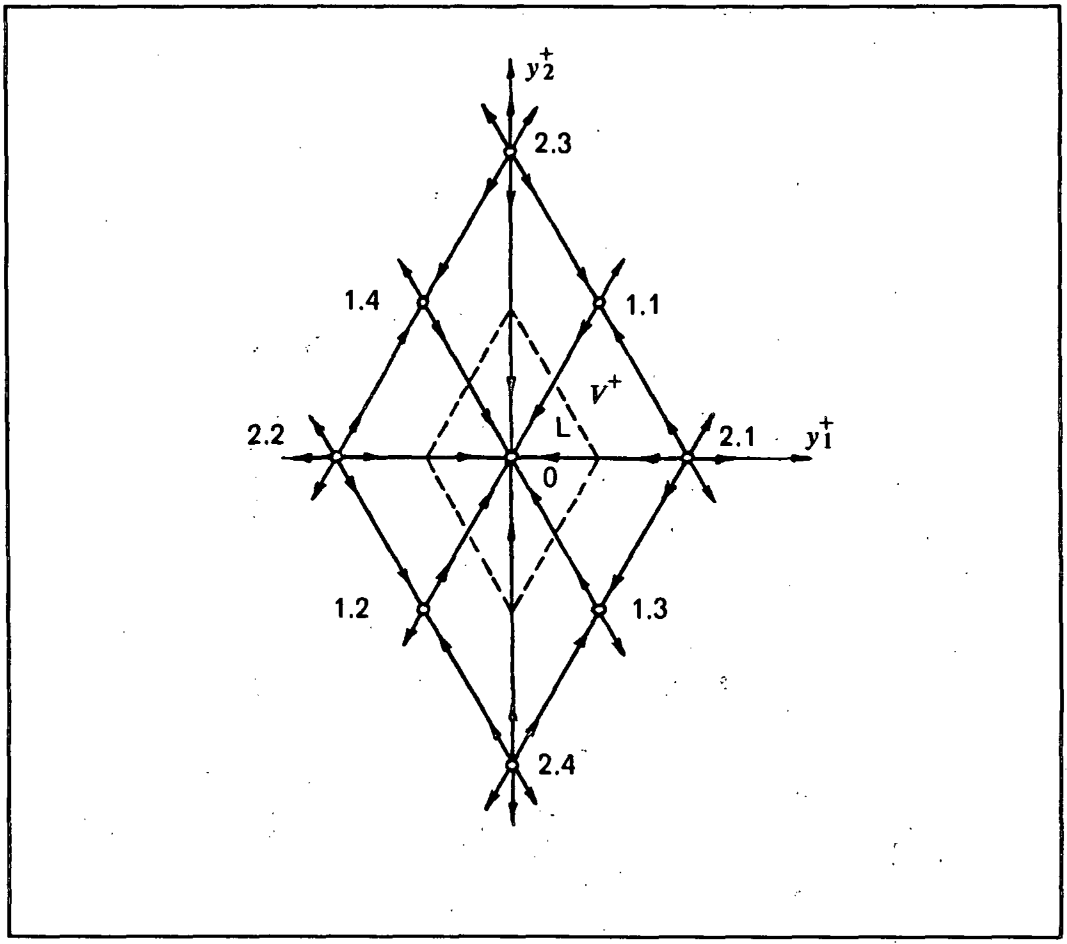



Figure 3.

If the conditions (16),(17) fulfilled, the coordinates of equilibriums 1.N1 of the saddle type [6] will then be:

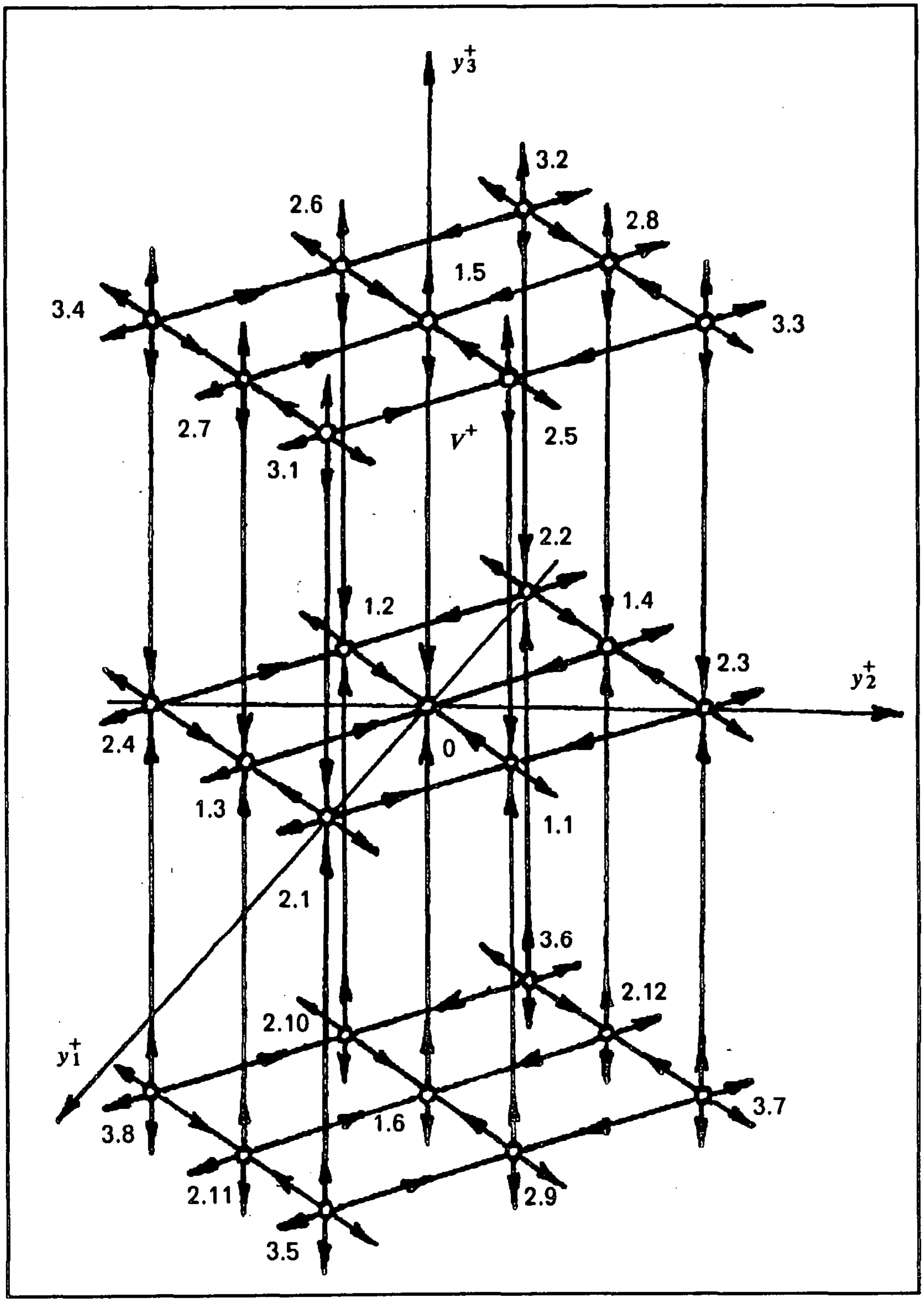
1.1 $y^+_1 = u^+/\lambda_1$, $y^+_2 = u^+/\lambda_2$ 1.2 $y^+_1 = -u^-/\lambda_1$, $y^+_2 = -u^-/\lambda_2$

1.3 $y^+_1 = u^+/\lambda_1$, $y^+_2 = -u^+/\lambda_2$ 1.4 $y^+_1 = -u^-/\lambda_1$, $y^+_2 = u^-/\lambda_2$

It may be shown that they are located ( Fig. 3 ) on the rays of unstable nodes 2.N2 having the coordinates

2.1 $y^+_1 = 2u^+/\lambda_1$, $y^+_2 = 0$ 2.2 $y^+_1 = -2u^-/\lambda_1$, $y^+_2 = 0$

2.3 $y^+_1 = 0$, $y^+_2 = 2u^+/\lambda_2$ 2.4 $y^+_1 = 0$, $y^+_2 = -2u^-/\lambda_2$



$y_3^+$
3.2
2.6
2.8
1.5
3.4
3.3
2.7
2.5
$V^+$
3.1
2.2
1.2
1.4
2.3
$y_2^+$
2.4
0
1.3
1.1
2.1
3.6
2.12
2.10
$y_1^+$
3.7
3.8
1.6
2.11
2.9
3.5


Figure 4.

For the complete system (5), these saddle-type equilibriums are with one- and two-dimention separatrix manifolds respectively. The boundaries of attracting region $V^+$ are assigned with straight line segments linking equilibriums 2. $N_2$ ( rays of unstable nodes). In the region $V^+$ ( Fig.3 ), by a dashed line there is indicated the region L of the system linearity, the region as the parameter $\beta_0$ grows is shrinking along the coordinates $y^+_1$ , $y^+_2$ towards the stabilizable equilibrium 0 that falls into the linearity region.

Further increase of the number s of positive eigenvalues in matrix A will give rise to the growth of dimension of phase space $Y^+$ whose structure at optimal-in stability control will be described through equlibriums:

- through a stable state O in the origin, and
- through unstable states of types 1. $N_1$, 2. $N_2$ ,.... s. $N_s$ corresponding to restrictions of one, two or s components in the cortrolled action vector.

The boundaries of region $V^+$ of attracting stabilizable equilibrium are assigned by separatrix surfaces containing unstable equlibriums of all types.

For extreme dimention s = 3 of the phase space $Y^+$ ( admitting a visual indication), its structure urder optimal-in-stability control is shown in Fig.4.

## CONCLUSION

The results obtained are interesting from the point of view of stability theory since they enable on the basis of stability "in small" of some small-dimention linearized subsystem to consider stability "in large" of some multi-dimention nonlinear system. Their practical significance lies in the capability of analytically synthesizing the algorithm of optimal-in-stability control on the basis of linearized description of an unstable object; the algorithm provides an optimal structure and parameters of the regulator. Admissable deviations of the structure and parameters of the regulator from the optimal ones, consideration of the effect produced by the nonlinearity of object's characteristics upon dynamics of a stabilizing system are described by known methods used in analysing the systems of automatic control.

Examples of such approach to problems of synthesizing and analysing specific systems are given in [ 7- 12] where there are corsidered the systems of stabilizing a magnetic suspension used nowadays in instrument and machine engineering and in high-speed land transport vehicles.

## REFERENCES

1. G.A. Stepaniants, N.S.Tararoshchenko, 1970. "On the Structure of Control Laws Ensuring Asymptotic Stability of Control System With An Unstable Object".Dokl.Akad.Nauk USSR (The USSR Scientific Academy Reports),193 (4):774-776.

2. G.A. Stepaniants, V.A.Shouvalov, 1986. "The Technique of Synthesizing the Laws for Stabilizing an Unstable Object with Limited Control". Priborostroenie (Instrument Engineering),26(6):15-19.

3. G.A. Stepaniants, 1985. Theory of Dinamic Systems, Moscow: Mashinostroenie, pp. 248.

4.R.Kalman,P.Fulb,M.Arbib,1971.Essay in the Mathematical Theory of System, Moscow: Mir, pp.400.

5. V.V.Rumyantsev, A.S.Oziraner, 1987. Stability and Stabilization of Motion In the Partial Variables, Moskow: Nauka, pp 253.

6.A.A.Andronov,A.A.Witt, S.E.Chaikin, 1959. Theory of Oscillations .Moscow: Fizmatgiz, pp 915.

7.V.S.Voronkov, O.D.Pozdeyev, 1980."Research of Systems for Stabilizing a Magnetic Bearing".Izvestia Akad.Nauk USSR Mekhanica Tverdogo Tela (The USSR Scientific Academy Reports The Solid Body Mechanics) No.4 pp 30-39.

8.V.S.Voronkov,1984."The Synthesis of a Stabilizing System for a Magnetic Suspension and Exsperimental Research of its Dynamics" Priborostroenie.27.(8):32-37.

9.V.S.Voronkov,O.D.Pozdeyev, 1987. "Magnetic Suspension Stabilization System Synthesis with Account Elastic Design." Priborostroenie. 30(2):61-65.

10.V.S.Voronkov, 1988. "Analyzing the Dynamics of Magnetic Suspension with Elastic Base".Priborostroenie.31(7):41-45.

11.V.S.Voronkov, 1989. "Synthesis of Magnetic Suspension Stabilizing System with Eddy Current Loop". Priborostroenie. 32,(5):51-56.

12.V.S.Voronkov, 1991. "Stabilizing a Shaft in Active Magnetic Bearings." Izvestia Acad. Nauk USSR Mekhanica Tverdogo Tela. No4, pp. 63-70 (all in Russian).