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STIFF AMB CONTROL USING AN $\mathcal{H}^\infty$ APPROACH

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Abstract

An important requirement in most practical AMB (active magnetic bearing) applications is: "Stiffness of the controlled mechanical parts, subjected to *unknown dynamic* disturbance forces or loads, should not be below a given value *for all relevant frequencies*." This requirement can be viewed as a *wide-band* disturbance attenuation problem in an $\mathcal{H}^\infty$ setting. This approach is particularly well-suited for applications where the *"worst case"* exciting frequency of disturbance forces must be considered.

The present contribution deals with trade-offs involved in the frequency domain. As in all control synthesis problems there are several *conflicting* requirements. A sample AMB problem is shown where the *achievable performance*, i.e. the worst-case compliance of the mechanical parts, is calculated numerically. Two basic ways of how *compromises* can be made and how several conflicting requirements can be incorporated in the $\mathcal{H}^\infty$ framework are considered.

1 Industrial Background

Electromagnetically supported milling spindles are examples for AMB application where stiff and precise suspension is indispensable [5][3]. The milling process induces wide-band cutting forces which may cause intolerable vibrations of the milling tool. The controller should provide a satisfactory "attenuation" of these vibrations.

2 A Sample AMB Problem

Consider the controlled mechanical system in figure 1 which stands for a simple electromagnetically supported *elastic* shaft. The system is assumed to be subjected to an *unknown* disturbance force $w(t)$ acting on the bottom mass m_1. Let $z(t)$ denote the displacement of m_1 caused by $w(t)$, and let $T_1(s)$ be the frequency-domain compliance of the bottom mass m_1: $z(s) = T_1(s)\, w(s)$.

For the time being, let the *objectives* of the controller C consist in *stabilizing* plant $\boldsymbol{P}$ and in maintaining dynamic *compliance* $T_1(s)$ *uniformly low*: $|T_1(i\omega)| \leq \alpha$, over all frequencies ω. The main goal of this paper is to seek such controllers and to gain an understanding of what happens if α is low.

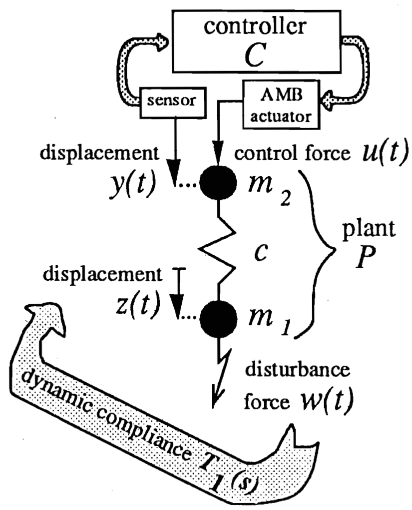

Figure 1: A sample AMB problem

The equations of motion are given by the following linear second order system:

$$\begin{cases} m_1\,\ddot{y} &= c\,(z-y) + u \\ m_2\,\ddot{z} &= c\,(y-z) + w \end{cases} \qquad (1)$$

Remark 2.1 For the sake of simplicity we do not consider the "negative stiffness" of the AMB, and we fix the parameter values to unity: $m_1 = m_2 = c = 1$.

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[3] Session 7 of this conference is also concerned with machine tool spindles.

Figure 2 shows the interconnections between plant $\boldsymbol{P}$ and controller C.

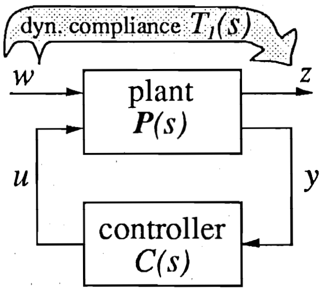

Figure 2: Interconnections

The corresponding algebraic equations in the Laplace–domain are:

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} \qquad (2)$$
$$u(s) = C(s)\, y(s)$$

The Laplace transformation of (1) gives:

$$\boldsymbol{P}(s) = \begin{bmatrix} \dfrac{s^2+1}{s^2(s^2+2)} & \dfrac{1}{s^2(s^2+2)} \\ \dfrac{1}{s^2(s^2+2)} & \dfrac{s^2+1}{s^2(s^2+2)} \end{bmatrix} \qquad (3)$$

It is a well–known fact that all closed–loop functions, of course including compliance T_1, can be expressed by *linear fractional maps* of controller C:

$$\begin{aligned} T_1(s) &= P_{11} + P_{12}C(1 - P_{22}C)^{-1}P_{21} \\ &\overset{\text{def}}{=:} \mathcal{F}_P(C(s)) \quad \text{(linear–fractional map)} \\ &= \frac{C(s) - (s^2+1)}{(s^2+1)C(s) - s^2(s^2+2)} \end{aligned} \qquad (4)$$

Definition 2.2 Let $\| T \|_\infty$ denote the *"worst case"* compliance defined by its peak value in the frequency response:

$$\| T \|_\infty \overset{\text{def}}{:=} \sup_{0 \le \omega < \infty} | T(i\omega) |$$

Design engineers do appreciate $\| \cdot \|_\infty$–norms, because they are handy and easy to measure. Now let us formulate the central problem of this paper:

Problem 2.3 Find *stabilizing* controllers C, such that the worst case compliance of the bottom mass m_1 is *"better"* than α, i.e. $\| T_1 \|_\infty \le \alpha$, where $\alpha > 0$ is a given user–specific value. What happens if α is low? Does an "optimal" controller $\breve{C}$ exist?

The first step towards the solution of problem 2.3 is to get an understanding of what a stabilizing controller actually is. The next section will clarify this point.

3 Closed–Loop Stability implies Interpolation Conditions on T_1

Let us start with an introductory question. Is the following procedure a legitimate way to obtain stabilizing controllers: "Choose an *arbitrary stable* compliance $T_1(s)$ and compute $C(s)$ using equation (4)"? We shall see that the answer is *no!* This section will provide mandatory constraints[4] on T_1.

Recall the basic definition of internal closed–loop stability:

Definition 3.1 First let us introduce two additional inputs v_1, v_2 and outputs u, y according to figure 3. The closed–loop $[\boldsymbol{P}, C]$ is termed *internally stable* [2] iff the augmented closed–loop transfer matrix $\boldsymbol{H}(\boldsymbol{P}, C)$ defined in equation (5) exists and belongs to $\mathcal{H}^\infty_{3\times 3}$; see next definition.

Definition 3.2 The *Hardy space* $\mathcal{H}^\infty$, or $\mathcal{H}^\infty_{m\times n}$ more precisely, consists of all complex functions of s (scalar–valued, vector–valued or matrix–valued according to m, n), which are *analytic* and *bounded* in $\{\Re e(s) > 0\}$. For the subset of *rational* functions arising from lumped systems an equivalent definition is: every entry $\boldsymbol{H}_{ij}(s)$ must be *proper* ($\boldsymbol{H}_{ij}(\infty)$ is finite) and *stable* (no poles in $\{\Re e(s) \ge 0\}$).

Definition 3.3 Let $\mathcal{S}(\boldsymbol{P})$ denote the *set* of all linear controllers $C(s)$ which *stabilize* $\boldsymbol{P}$.

Here is the (3×3) transfer matrix $\boldsymbol{H}(\boldsymbol{P}, C)$:

$$\begin{bmatrix} z \\ u \\ y \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix} \qquad (5)$$

Using the one–to–one correspondence between controller C and compliance T_1 in equation (4), we compute $\boldsymbol{H}$ as a function of T_1:

[4] An alternative approach [2] is coprime factorization of P_{22}.

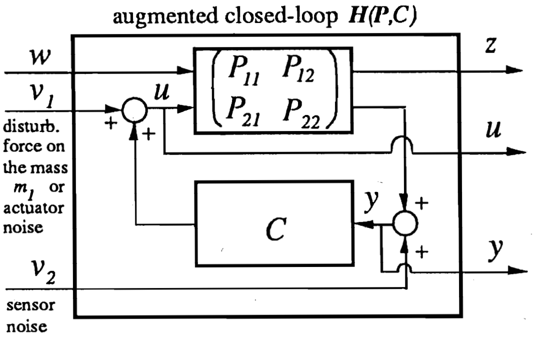

Figure 3: Diagram for stability definition

$$\text{1. col. of } \boldsymbol{H}: \begin{bmatrix} T_1 \\ s^2(s^2+2)T_1 - (s^2+1) \\ (s^2+1)T_1 - 1 \end{bmatrix}$$

$$\text{2. col. of } \boldsymbol{H}: \begin{bmatrix} (s^2+1)T_1 - 1 \\ s^2(s^2+2)\left((s^2+1)T_1 - 1\right) \\ (s^2+1)\left((s^2+1)T_1 - 1\right) \end{bmatrix}$$

$$\text{3. col. of } \boldsymbol{H}: \begin{bmatrix} s^2(s^2+2)T_1 - (s^2+1) \\ s^2(s^2+2)\left(s^2(s^2+2)T_1 - (s^2+1)\right) \\ s^2(s^2+2)\left((s^2+1)T_1 - 1\right) \end{bmatrix}$$

In fact, $\boldsymbol{H}$ is an *affine* function of T_1:

$$H_{ij}(s) = p_{ij}(s)\,T_1(s) + q_{ij}(s) \qquad (6)$$

where $p_{ij}(s)$ and $q_{ij}(s)$ are given *polynomials* in s. Therefore, $\boldsymbol{H}$ and T_1 share the same poles, except possibly for $s=\infty$. A *necessary* condition for internal stability is $T_1 \in \mathcal{H}^\infty$, but of course this is insufficient to meet the boundedness of $\boldsymbol{H}(\infty)$. Expand $T_1 \in \mathcal{H}^\infty$ as Laurent series around infinity:

$$T_1(s) = a_0 + \frac{a_{-1}}{s} + \frac{a_{-2}}{s^2} + \frac{a_{-3}}{s^3} + \dots \qquad (7)$$

Insert (7) into (6) in order to get the asymptotic Laurent expansions of $\boldsymbol{H}_{ij}(s)$. Obviously, all positive powers in s have to vanish, which leads to the following conditions for the coefficients $a_0 \dots a_{-7}$:

$$\left\{ \begin{array}{rclrcl} a_0 &=& 0, & a_{-1} &=& 0 \\ a_{-2} &=& 1, & a_{-3} &=& 0 \\ a_{-4} &=& -1, & a_{-5} &=& 0 \\ a_{-6} &=& 2, & a_{-7} &=& 0 \end{array} \right. \qquad (8)$$

The coefficients $a_{-8}, a_{-9}, \dots$ *freely* depend on the controller C.

Statement 3.4 *The closed loop* $[\boldsymbol{P}, C]$ *is internally stable iff compliance* $T_1(s)$ *is stable and meets the interpolation conditions* (8). *We will call* T_1 *to be admissible iff these two conditions are met. Section 3 established the following equivalence relation:* C *stabilizing* $\iff$ $\boldsymbol{H}(\boldsymbol{P},C) \in \mathcal{H}^\infty_{3\times 3}$ $\iff$ T_1 *admissible*

4 The "forbidden" controller C_0

Set $T_1(s) \equiv 0$ which *violates* the interpolation conditions (8)! Equation (4) gives:

$$C_0(s) = s^2 + 1 \quad \notin \mathcal{S}(\boldsymbol{P})$$

Figure 4 helps to clarify the behaviour of C_0. Of

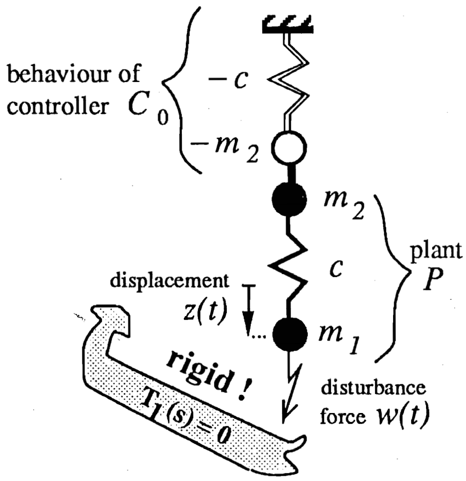

Figure 4: Behaviour of controller C_0

course, C_0 is neither stabilizing nor is it implementable (differentiators do not exist physically). Let us take a look at the corresponding closed–loop matrix $\boldsymbol{H}_0 := \boldsymbol{H}(\boldsymbol{P}, C_0)$:

$$\boldsymbol{H}_0(s) = \begin{bmatrix} 0 & -1 & -s^2-1 \\ -s^2-1 & -s^4-2s^2 & -s^2(s^2+1)(s^2+2) \\ -1 & -s^2-1 & -s^4-2s^2 \end{bmatrix} \notin \mathcal{H}^\infty_{3\times 3}$$

Most of the $[\boldsymbol{H}_0]_{ij}$ are unbounded for $s \to \infty$: *high frequency* sensor (actuator) noise causes actuator (sensor) *saturation*. So, most of the functions H_{ij}

conflict with a low compliance $\| T_1 \|_\infty$. Note the practical meaning of each of the H_{ij}: we already introduced $H_{11} = T_1$ which is the local compliance of the *bottom* mass m_1. $T_2 := H_{32}$ denotes the local compliance of the *top* mass m_2.

The following statement 4.1 will fully make sense to the reader at the latest when he has finished reading this paper!

Statement 4.1

- *There are stabilizing controllers C_j which permit an arbitrary low worst case compliance $\| T_1 \|_\infty$ of the bottom mass m_1.*
- *There are also stabilizing controllers $\tilde{C}_j$ which permit an arbitrary low worst case compliance $\| T_2 \|_\infty$ of the top mass m_2.*
- *However, controllers which permit both at the same time do not exist.*

In fact, the optimization problem "min $\| T \|_\infty$" turned out to be a degenerate *boundary interpolation* problem at $s = \infty$ (see [1], [4]), because it lacks any *controller bandwidth limitation.* In order to take this essential aspect into consideration we now present two approaches.

5 First approach

The first approach is somehow indirect: we must prevent the closed–loop poles from "migrating too far to the left". This can be achieved by reducing the allowed pole region to a disk by means of a bilinear map, see figure 5. Parameter β governs the size of this disk. As β approaches 1, the disk grows to cover the whole left half plane $\{\Re e(s) < 0\}$. So, the modified version of problem 2.3 is:

Problem 5.1 Define the disk $\mathcal{D}_\beta := \{|s + (1-\beta)^{-1}| < (1-\beta)^{-1}\}$. A compliance function $T_1(s)$ will be called *β-admissible*, if *all* of its poles are located in $\mathcal{D}_\beta$ *and* if it meets the interpolation conditions (8). Now, find a compliance function $\check{T}_\beta$ which solves the following Min–Max problem:

$$\inf_{T_1\ \beta\text{-admissible}} \left\{ \sup_{s\, \in\, \partial\mathcal{D}_\beta} |T_1(s)| \right\}$$

The optimal value[5] is denoted by $\check{\alpha}(\beta)$.

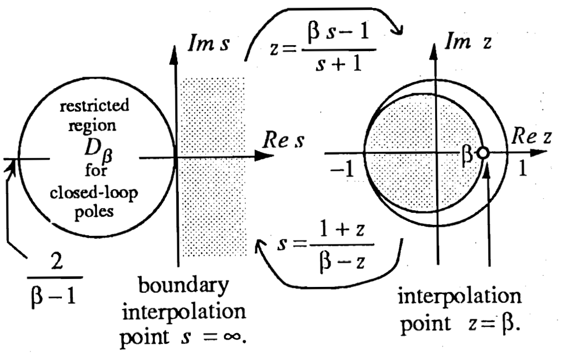

Figure 5: Bilinear map

Here is a brief outline of the standard "machinery" [2][6] we used to compute the solution of problem 5.1:

- Transform the interpolation conditions (8) in the z–plane. The new interpolation point is $z_\star = \beta$. The function values up to the seventh derivative are prescribed at $z_\star$.
- Set up the "model matching" problem inf $\| T_a - QT_b \|_\infty$. The Blaschke product $(z-\beta)^8/(1-z\beta)^8$ is a good choice for T_b.
- Set up the "Nehari" problem inf $\| R - X \|_\infty$.
- Solve the Nehari problem (by solving Lyapunov equations and eigenvalue problems).
- Backsubstitute and use the reverse bilinear map. Finally use (4) to obtain the corresponding controller $\check{C}_\beta$.

Example 5.2 Choose $\beta = 0.6$. Thus the closed–loop pole locations are restricted to the disk $\{|s + 2.5| < 2.5\}$. For this value of β we obtained the following numerical solution:

$$\| \check{T}_{\beta=0.6}(s) \|_\infty = \check{\alpha}(0.6) \approx 1.727$$

The resulting controller $\check{C}_{\beta=0.6}$ is:

$$\approx \frac{-2.55798s^3 + 0.7724s^2 - 4.68039s - 2.14526}{0.001s^3 + 0.01979s^2 - 2.38978s + 1.55966}$$

Note that compliance $|\check{T}_{\beta=0.6}(s)|$ is *constant* on the circle $\{|s + 2.5| = 2.5\}$.

[5] Note that by the maximum modulus theorem $\| \check{T} \|_\infty \le \check{\alpha}$. In fact, the right hand side equals the left due to the "all pass" property of solution $\check{T}_\beta$.

[6] Recently, a new state space approach appeared in literature, see [Doyle, Glover, Khargonekar, Francis, IEEE 8, 1989]. This new method avoids the cumbersome backsubstitution.

Remark 5.3 Note that the order of $\check{C}$ is $n-1 = 3$. This is a well-known result in $\mathcal{H}^\infty$ control theory.

Figure 6 shows the set of solutions as β varies. We see that the optimum value of the worst-case compliance decreases to zero as $\beta \to 1$; at the same time the gain of the corresponding controllers $\check{C}_\beta$ keeps increasing.

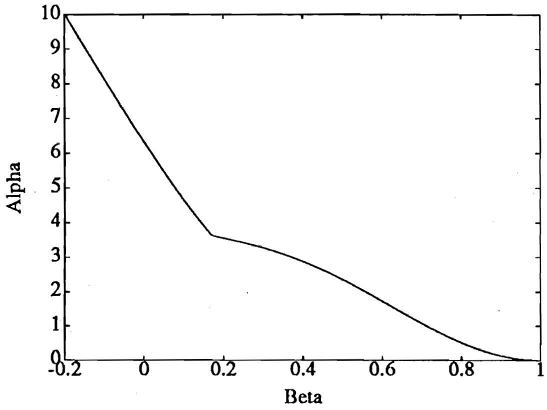

Figure 6: Numerical evaluation of $\check{\alpha}(\beta)$

6 Second approach

One disadvantage of the first approach is: parameter β of the bilinear map offers only insufficient control of the "size" of the functions $H_{ij}(s)$ conflicting with a low compliance $\| T_1 \|_\infty$. An interesting step towards a remedy would be a *quantitative* version of statement 4.1, namely: we already know that *no* controller C exists which permits *arbitrary low* values of *both* $\| T_1 \|_\infty$ *and* $\| T_2 \|_\infty$. But what about minimizing one of those while keeping the other one under a given tolerable bound ϱ? For the sake of simplicity we now don't care about sensor noise and actuator saturation.

Problem 6.1 Find a *stabilizing* controller C_ϱ which minimizes $\| T_1 \|_\infty$ under the *constraint* $\| T_2 \|_\infty \le \varrho$, where $\varrho > 0$ is given.

Unfortunately, problem 6.1 is a very intricate one, and it cannot be solved in this setting. We now give a solvable problem slightly related to 6.1:

Problem 6.2 Define the dynamic "cross"-compliances $T_{12} := H_{12}$ and $T_{21} := H_{31}$. Note that $T_{12} = T_{21}$ because of the *reciprocity* of non-gyroscopic mechanical systems. Define a diagonally ϱ-weighted compliance matrix $\boldsymbol{W}_\varrho(s)$ according to figure 7. Our optimization problem is[7]:

$$\inf_{T_1 \text{ admissible}} \Bigg\| \underbrace{\begin{bmatrix} \varrho^2 T_1 & T_{12} \\ T_{12} & \varrho^{-2} T_2 \end{bmatrix}}_{=: \boldsymbol{W}_\varrho(s)} \Bigg\|_\infty$$

Recall that the $\| \cdot \|_\infty$-norm for matrix-valued transfer functions is defined by its maximum singular value over the $i\omega$ frequency axis.

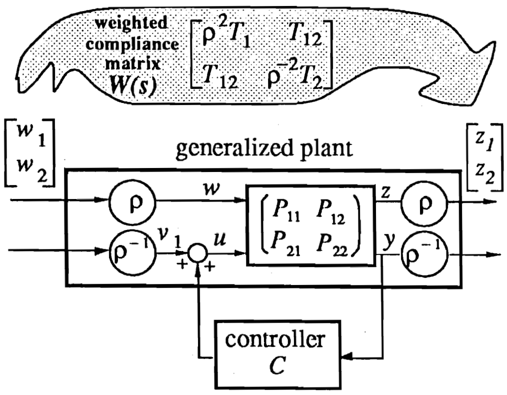

Figure 7: Seeking compromises by weighting

A careful analysis [3] of problem 6.2 shows that the infimum over all *admissible* T_1's is not achieved[8] but it is approached by a sequence of admissible T_1's. In order to compute the limit $\check{T}_1$ of this sequence which is not admissible in the sense of statement 3.4 we will temporarily relax the interpolation conditions. For the moment let us only insist on the first four conditions on $a_0 \ldots a_{-3}$. Then the corresponding "model matching" problem is easily shown [3] to be:

$$\check{\alpha}(\varrho) := \inf_{Q \,\in\, \mathcal{H}^\infty_{1\times 1}} \Big\| \underbrace{\boldsymbol{T}_a - \boldsymbol{T}_b\, Q\, \boldsymbol{T}_c}_{= \boldsymbol{W}_\varrho(s)} \Big\|_\infty$$

$\boldsymbol{T}_a \in \mathcal{H}^\infty_{2\times 2}$, $\boldsymbol{T}_b \in \mathcal{H}^\infty_{2\times 1}$ and $\boldsymbol{T}_c \in \mathcal{H}^\infty_{1\times 2}$ are as follows:

$$\boldsymbol{T}_a(s) := \begin{bmatrix} \dfrac{\varrho^2\,(s^2+2s+2)}{(s^2+s+1)^2} & \dfrac{1}{(s^2+s+1)^2} \\[2ex] \dfrac{1}{(s^2+s+1)^2} & \dfrac{\varrho^{-2}\,(s^2+1)}{(s^2+s+1)^2} \end{bmatrix}$$

[7] Recall that T_1 uniquely determines T_2 and T_{12}.
[8] The reason for this is that we did not incorporate H_{23} in the optimization problem. Note that the order of the polynomial $p_{23}(s)$ in equation (6) is the highest and comes to 8.

$$T_c(s) := T_b^T(s) := \left[\begin{array}{cc} \frac{\varrho}{(s+1)^2} & \frac{\varrho^{-1}(s^2+1)}{(s+1)^2} \end{array} \right]$$

We solved this model matching problem by using the standard technique [2] (reduce it to the "4–block" problem which can be iteratively solved).

Example 6.3 Choose $\varrho = 0.95$. We obtained the following numerical solution:

$$\| \breve{W}_{\varrho=0.95}(s) \|_\infty = \breve{\alpha}(0.95) \approx 1.7028$$

$$\| \breve{T}_1 \|_\infty \approx 1.562, \quad \| \breve{T}_2 \|_\infty \approx 1.537$$

The corresponding controller $\breve{C}_{\varrho=0.95}$ is:

$$\approx \frac{-1.5368s^3 + 0.6490s^2 - 2.5368s - 1.1543}{-1.5368s + 0.6490}$$

Of course, $\breve{C}$ has to be "replaced" by a *proper* stabilizing controller. This can be done either prior to this stage (e.g. by the method of section 5) or afterwards (e.g. by "cutting off" Q, see [3]). The limitation of the controller bandwidth further reduces the achievable performance, see section 5.

Figure 8 shows the set of trade-off solutions as the weighting parameter ϱ varies. Choosing ϱ large implies a low worst–case compliance of mass m_1 at the expense of a large worst–case compliance of mass m_2. Choosing ϱ small reverses the situation. Statement 4.1 is thus fully confirmed.

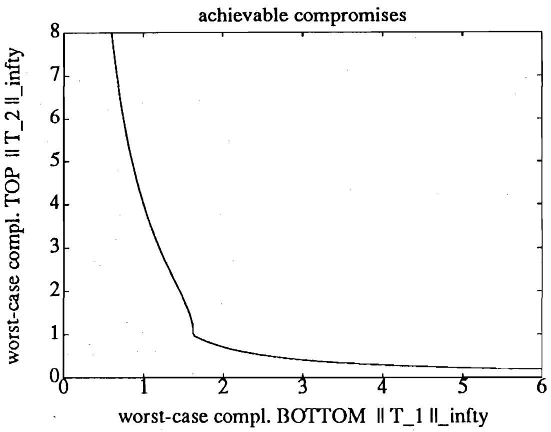

Figure 8: $\varrho \mapsto (\| \breve{T}_1 \|_\infty, \| \breve{T}_2 \|_\infty)$: Parametric plot for ϱ sweeping from $0.35 \ldots 2$.

7 Conclusions and Outlook

This paper gives some insight into how modern $\mathcal{H}^\infty$ control theory can be used to treat *wide–band stiffness requirements* in AMB applications.

We are convinced that $\mathcal{H}^\infty$ approaches are a *powerful* and *extremely versatile* tool to determine the achievable performance in AMB feedback systems, and to design the corresponding controllers. Of course, one has to learn how to use this new tool and how to choose *"appropriate"* $\mathcal{H}^\infty$ design criteria. The meaning of "appropriate" inherently depends on the practical application.

Some future problems we want to tackle are robustness aspects in AMB control and design of positive real controllers using $\mathcal{H}^\infty$ optimality criteria. It should be intuitively clear that "highly stiff" AMB controllers have only poor "robustness properties". Recall that we identified the "stiffest" controller C_0 with a *negative* mass and a *negative* spring. Positive real controllers are exactly the opposite: they behave as an interconnection of *positive* dampers, masses and springs. We expect this conservatism to improve the "robustness properties" at the expense of "lower performance".

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