2nd International Symposium on Magnetic Beaing
July 12-14, 1990, Tokyo, Japan

A DECOMPOSITION OF THE JEFFCOTT ROTOR

D. FERMENTAL, E. CUSSON, P. LAROCCA
The Charles Stark Draper Laboratory, Inc.
555 Technology Square, Cambridge, MA, USA

Abstract

This report provides a novel representation of a magnetically suspended rotor consisting of a single inertia, supported on a massless shaft exhibiting stiffness and damping (the Jeffcott model). The state variables are chosen in a way that allows the plant to be transformed into subsections. These subsections independently account for translation, rotation and bending. The report also derives observability and controllability matrices for the Jeffcott rotor.

1 Introduction

This effort provides representations of a magnetically levitated rotor consisting of an inertia on a shaft. The inertia is not centered and the magnetic forces act on the shaft ends where the system sensors are also located. The study, which extends the work of Johnson[1] and McCallum[2] includes two plant models. In one model the shaft is rigid. In the second, the Jeffcott model, the shaft bends. In both, the shaft is taken as massless, which allows the second model to be treated as an embellishment of the first. The development assumes an isotropic shaft with internal damping only. The result is a plant that becomes unstable, and remains so, above the first critical speed, a spin rate corresponding to the undamped natural frequency of plant translational oscillations when the spin rate is zero.

The plant equations used for the Jeffcott model are identical to those derived in McCallum[2]. However, the plant state vector is defined in a different way. The equations account for both translation and rotation of the shaft and include gyroscopic effects.

Unlike the work of McCallum, it is assumed in this document that the magnetic bearings are compensated. This is justified since the gap compensation must be part of a design effort in any case. Here, it is assumed local feedback at each bearing removes the position dependence, and thus, the negative stiffness element associated with bearing magnetic field.

Johnson[1], considering translational motion only, showed that the Jeffcott model is both controllable and observable at all spin rates. McCallum[2] extended this result to include radial rotation and its associated gyroscopic effects, but provides no determinant for either the controllability matrix or the observability matrix. These are supplied in closed form in this report.

2 Model References

Initially the equations are cast to include linearized terms that represent uncompensated magnetic bearings. Later it will be assumed that the bearings are compensated, and the magnetic terms will be dropped from the analysis.

The rotor is taken to be an inertia on a shaft supported at its ends. The inertia is at a distance, L1, from the leftmost support, and a distance, L2, from the rightmost support. Figure 1 presents a side view and shows vertical forces. Horizontal forces can be shown in a like manner. Axial motion is not considered.

The spin axis of the rotor is taken to be along the X axis with clockwise spin being a positive angular rotation, ω_x. Vertical motion is along the Y axis, and horizontal motion is along the Z axis. The yaw angle, ξ_1, is about the Y axis, and the pitch angle, ξ_2, is about the Z axis.

In Figure 2 the spring constant for the magnetic bearing at bearing 1 in the vertical direction is -Ky1 and the vertical force, Fsy1, acting on the shaft satifies

$$Fsy1 = Fy1 + Ky1 * Y1$$

where Y1 is the vertical displacement of the shaft end, and Fy1 is a plant input. Note that the force on the shaft due to the bias field Ky1*Y1 increases with displacement. The forces Fsy2, Fsz1 and Fsz2 are similar in form.

The spring, dashpot combination (K1,B1) is used to model the shaft stiffness and damping between the inertia and the leftmost support, while the combination (K2,B2) models the other section. Since both stiffness and damping are inversely proportional to length they can be expressed as a ratio

$$\frac{K1}{B1} = \frac{K2}{B2} = S_o.$$

Figure 2 shows the spring-dashpot combination for the shaft at the leftmost support from the rear. The couplet (Y1,Z1) defines the position of the shaft while (Ys1,Zs1) is the end of the shaft without bending. If the pitch angle is small.

$$Ys1 = Yr - L1 * \xi_2$$

The difference Ys1 - Y1 is due to the deflection of the shaft and is represented by the variable DY1. DY2, DZ1, and DZ2 are defined in a similar manner. The inertia of the plant about the X axis, is J_a. The inertia about the Y axis and the Z axis, is J_r. It is assumed in the analysis that the pitch and yaw angles are sufficiently small and that the values of J_a and J_r are independent of plant motion.

3 The Jeffcott Model

From Fig. 1 the forces acting on the rotor in the vertical direction are Fsy1 and Fsy2. These forces tend to move the rotor both in translation and rotation. The equations for the translation motion, neglecting gravity is

$$Fsy1 + Fsy2 = \ddot{M} * Yr$$

where M is the rotor mass. The equation for the rotational motion is

$$L2 * Fsy2 - L1 * Fsy1 = Jr * \ddot{\xi}_2 - Ja * \omega_x * \dot{\xi}_1$$

where $Ja * \omega_x * \dot{\xi}_1$ is the gyroscopic torque caused by changing the yaw angle as the rotor spins about the X axis with angular velocity ω_x. If the rotor is balanced the equations can be written as

$$\begin{aligned}
\ddot{Y_r} &= \frac{(Ky1 + Ky2) * Yr}{M} - \frac{(L1 * Ky1 - L2 * Ky2) * \xi_2}{M} \\
&+ \frac{Fy1 + Fy2}{M} \\
\ddot{Zr} &= \frac{(Kz1 + Kz2) * Zr}{M} + \frac{(L1 * Kz1 - L2 * Kz2) * \xi_1}{M} \\
&+ \frac{Fz1 + Fz2}{M} \\
\ddot{\xi_2} &= \frac{(L1^2 * Ky1 + L2^2 * Ky2) * \xi_2}{Jr} \\
&- \frac{(L1 * Ky1 - L2 * Ky2) * Yr}{Jr} \\
&\frac{-L1 * Fy1 + L2 * Fy2 + Ja * \omega_x * \dot{\xi}_1}{Jr} \\
\ddot{\xi_1} &= \frac{(L1^2 * Kz1 + L2^2 * Kz2) * \xi_1}{Jr} \\
&+ \frac{(L1 * Kz1 - L2 * Kz2) * Zr}{Jr} \\
&+ \frac{L1 * Fz1 - L2 * Fz2 - Ja * \omega_x * \dot{\xi}_2}{Jr}.
\end{aligned}$$

These equations coupled with

$$Y1 = Yr - L1 * \xi_2; \quad Y2 = Yr + L2 * \xi_2$$

$$Z1 = Yr + L1 * \xi_1; \quad Z2 = Yr - L2 * \xi_1$$

describe the rigid plant. The state variables are

$$[Y_r, \xi_2, Z_r, \xi_1, \dot{Y}_r, \dot{\xi}_2, \dot{Z}_r, \dot{\xi}_1].$$

Figure 2 shows the deflection of the shaft as seen from the leftmost support or bearing 1. The force acting to bend the shaft, Fs1, is the vector sum of Fsy1 and Fsz1. The deflection of the shaft at the bearing, DR1, satisfies

$$Fs1 = K1 * DR1 + B1 * \dot{DR}1.$$

The forces, Fsy1, and, Fsz1, can be written as

$$Fsy1 = K1 * Cos(\theta) * DR1 + B1 * Cos(\theta) * \dot{DR}1$$

$$Fsz1 = K1 * Sin(\theta) * DR1 - B1 * Sin(\theta) * \dot{DR}1.$$

Since,

$$\begin{aligned} DY1 &= Cos(\theta) * DR1 \\ DZ1 &= Sin(\theta) * DR1 \\ D\dot{Y}1 &= Cos(\theta) * D\dot{R}1 - \omega_x * Sin(\theta) * DR1 \\ D\dot{Z}1 &= Sin(\theta) * D\dot{R}1 - \omega_x * Cos(\theta) * DR1 \end{aligned}$$

we can write the force equations as

$$\begin{aligned} Fsy1 &= K1 * DY1 + B1 * (D\dot{Y}1 + \omega_x * DZ1) \\ Fsz1 &= K1 * DZ1 + B1 * (D\dot{Z}1 - \omega_x * DY1) \\ Fsy2 &= K1 * DY2 + B1 * (D\dot{Y}2 + \omega_x * DZ2) \\ Fsz2 &= K1 * DZ2 + B1 * (D\dot{Z}2 + \omega_x * DY2). \end{aligned}$$

Crandall[3] speaks of the cross terms in these equations as being quasi— gyroscopic. They provide the mechanism for coupling energy from the rotor spin into rotation of the shaft deflection, the shaft whirl.

These results can be recast as equations for the additional state variables [DY1, DZ1, DY2, DZ2] thus,

$$D\dot{Y}1 = \frac{Fy1}{B1} + \frac{Ky1 * Y1}{B1} - S_o * DY1 - \omega_x * DZ1$$

$$D\dot{Z}1 = \frac{Fz1}{B1} + \frac{Kz1 * Z1}{B1} - S_o * DZ1 - \omega_x * DY1$$

$$D\dot{Y}2 = \frac{Fy2}{B2} + \frac{Ky2 * Y2}{B1} - S_o * DY2 + \omega_x * DZ2$$

$$D\dot{Z}1 = \frac{Fz2}{B2} + \frac{Kz2 * Z2}{B2} - S_o * DZ2 + \omega_x * DY2.$$

The state vector $[\mathbf{x}]^t$ for a shaft with bending can now be defined as

$$[Y_r, \xi_2, Z_r, \xi_1, \dot{Y}_r, \dot{\xi}_2, \dot{Z}_r, \dot{\xi}_1, D_{Y1}, D_{Z1}, D_{Y2}, D_{Z2}].$$

With no mass imbalance, the forces acting on the rotor are

$$Fsy1 = Fy1 + Ky1 * Y1; \quad Fsy2 = Fy2 + Ky2 * Y2$$

$$Fsz1 = Fz1 + Kz1 * Z1; \quad Fsz2 = Fz2 + Kz2 * Z2.$$

For compensated bearings, bearings with drives so designed that forces on the shaft due to the end positions are removed by local feedback, the stiffness constants [Ky1, Ky2, Kz1, Kz2] are zero and the plant input vector is defined as

$$[\mathbf{u}]^t = [F_{Y1}, F_{Z1}, F_{Y2}, F_{Z2}]$$

The output vector is defined as

$$[\mathbf{y}]^t = [Y_1, Y_2, Z_1, Z_2].$$

where

$$Y1 = Yr - L1 * \xi_2 + DY1; \quad Y2 = Yr + L2 * \xi_2 + DY2$$

$$Z1 = Zr + L1 * \xi_1 + DZ1; \quad Z2 = Zr - L2 * \xi_1 + DZ2.$$

$$\begin{aligned} \dot{Y}1 &= \frac{K1 * Yr}{B1} + \dot{Y}r + \omega_x * (Zr + L1 * \xi_1 - Z1) \\ &\quad - \frac{K1 * L1 * \xi_2}{B1} - L1 * \dot{\xi}_2 \\ &\quad + \frac{(Ky1 - K1) * Y1}{B1} + \frac{Fy1}{B1} \\ \dot{Y}2 &= \frac{K2 * Yr}{B2} + \dot{Y}r + \omega_x * (Zr - L2 * \xi_1 - Z2) \\ &\quad + \frac{K2 * L2 * \xi_2}{B2} + L2 * \dot{\xi}_2 \\ &\quad + \frac{(Ky2 - K2) * Y2}{B2} + \frac{Fy2}{B2} \\ \dot{Z}1 &= \frac{K1 * Zr}{B1} + \dot{Z}r + \omega_x * (-Yr + L1 * \xi_2 + Z2) \\ &\quad + \frac{K1 * L1 * \xi_1}{B1} + L1 * \dot{\xi}_1 \\ &\quad + + \frac{(Kz1 - K1) * Z1}{B1} \frac{Fz1}{B1} \\ \dot{Z}2 &= \frac{K2 * Zr}{B2} + \dot{Z}r + \omega_x * (-Yr - L2 * \xi_2 + Y2) \\ &\quad - \frac{K2 * L2 * \xi_1}{B2} - L2 * \dot{\xi}_1 \\ &\quad + \frac{(Kz2 - K2) * Z2}{B2} + \frac{Fz2}{B2} \end{aligned}$$

The translational forces are

$$Fyr = Fsy2 + Fsy2; \quad Fzr = Fsz1 + Fsz2$$

and the torques are

$$\begin{aligned} Te2 &= -L1 * Fsy1 + L2 * Fsy2 \\ Te1 &= L1 * Fsz1 - L2 * Fsz2. \end{aligned}$$

These results completely describe the Jeffcott model with internal damping. Fig. 3 shows a Block diagram of the model. The subsection for bending, translational and rotation can be easily identified.

4 Normalization

The model is normalized in the following way. Let

$$\begin{aligned} 1 \text{ unit of length} &= \text{L1 meters} \\ 1 \text{ unit of time} &= \tfrac{1}{S_0} \text{ seconds} \\ 1 \text{ unit of mass} &= M \text{ kg} \end{aligned}$$

Therefore,

$$1 \text{ unit of force} = M^*\text{L1}^*S_0^2 \text{ newtons}.$$

The damping factor associated with the shaft between the mass and bearing 1 is B1. It's normalized value is $\frac{1}{v}$ where

$$\frac{1}{v} = \frac{B1}{M * S_0}.$$

Since $\frac{K1}{B1} = S_0$, where K1 is the stiffness constant for the same shaft section, the normalized value of K1 is $\frac{1}{v}$ as well. The inertia, Jr, in normalized terms becomes

$$\frac{1}{m} = \frac{Jr}{M * L1^2}.$$

To complete the normalization, three other ratios are required

$$\begin{aligned} \text{r} &= \frac{L2}{L1} = \frac{K2}{K1}\frac{B2}{B1} \\ \text{w} &= \frac{\omega_x}{S_0} \\ a &= \frac{Ja}{Jr} * \text{w}. \end{aligned}$$

For the system equations in the standard state space form

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u}. \end{aligned}$$

The normalized plant matrices are:

$$[A] =$$

$$\begin{bmatrix} 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&a&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&-a&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&-1&-w&0&0 \\ 0&0&0&0&0&0&0&0&w&-1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&-1&-w \\ 0&0&0&0&0&0&0&0&0&0&w&-1 \end{bmatrix}$$

$$[B] =$$

$$\begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&1&0 \\ -m&0&mr&0 \\ 0&1&0&1 \\ 0&m&0&mr \\ v&0&0&0 \\ 0&v&0&0 \\ 0&0&rv&0 \\ 0&0&0&rv \end{bmatrix}$$

$$[C] =$$

$$\begin{bmatrix} 1&-1&0&0&0&0&0&0&1&0&0&0 \\ 1&r&0&0&0&0&0&0&0&1&0&0 \\ 0&0&1&1&0&0&0&0&0&0&1&0 \\ 0&0&1&-r&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$[D] =$$

$$\begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

5 Controllability

The system controllability matrix is

$$[C_n] = [B][A * B][A^2 * B].$$

In normalized form it is equal to

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -m & 0 & mr & 0 & 0 & am & 0 & -amr \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m & 0 & -mr & am & 0 & -amr & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -m & 0 & mr & 0 & 0 & am & 0 & -amr & a^2m & 0 & -a^2m & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & -mr & am & 0 & -amr & 0 & 0 & a^2m & 0 & a^2m \\ v & 0 & 0 & 0 & -v & -vw & 0 & 0 & -v^2 & -v^2w & 0 & 0 \\ 0 & v & 0 & 0 & vw & -v & 0 & 0 & v^2w & -v^2 & 0 & 0 \\ 0 & 0 & rv & 0 & 0 & 0 & -rv & -rvw & 0 & 0 & -rv^2 & -rv^2w \\ 0 & 0 & 0 & rv & 0 & 0 & rvw & -rv & 0 & 0 & rv^2w & -v^2 \end{bmatrix}$$

and

$$det[C_n] = (mv(r+1))^4 * r^2 * (1+w^2)^3 * (1+(w-a)^2).$$

Thus, det$[C_n]$ is positive, and the plant is controllable, for all choices of plant parameters, except r = -1. This simply means the plant requires two separate bearings to be controllable.

6 Observability

The observability matrix is

$$[O_b] = [[C][C * B][CB^2]]^t.$$

In normalized coefficients $[O_b]$ =

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -r & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & -w & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & r & 0 & 0 & w & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -w \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -r & 0 & 0 & w & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & (1-w^2) & 2w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & ar & -2w & (1-w^2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & (1-w^2) & 2w \\ 0 & 0 & 0 & 0 & 0 & -ar & 0 & 0 & 0 & 0 & -2w & (1-w^2) \end{bmatrix}$$

and

$$det[O_b] = (r+1)^2 * (w^2+1)^2 * [(r+1)^2 * (w^2+1)^2 + a^2 * (w * (r-1) + (r+1))^2].$$

Thus, det$[O_b]$ is positive and the plant is observable for all choices of plant parameters, except again r = -1.

7 Conclusion

The observability and controllability matrix for the Jeffcott model shows that the system is both observable and controllable for all rotation speeds. The model can be been decomposed into a translational, rotational and bending subsection. An attractive feature of this model is, independent control laws can be designed for each subsection and filtering requirements can be tailored to meet their particular needs.

References

[1] Johnson B. G., Active Control of a Flexible Rotor, S.M. Theses, Massachusetts Institute of Technology, September 1986, CSDL-T-864.

[2] McCallum D. C. Dynamic Modelling and Control of a Magnetic Bearing-Suspended Rotor System, S.M. Thesis, Massachusetts Institute of Technology, May 1988, CSDL-T-995.

[3] Crandall S. H., Physical Explanations of the Destabilizing Effect of Damping in Rotating Parts, Rotordynamic Instability Problems in High Performance Turbomachinery, NASA CP 2133, May 1980.

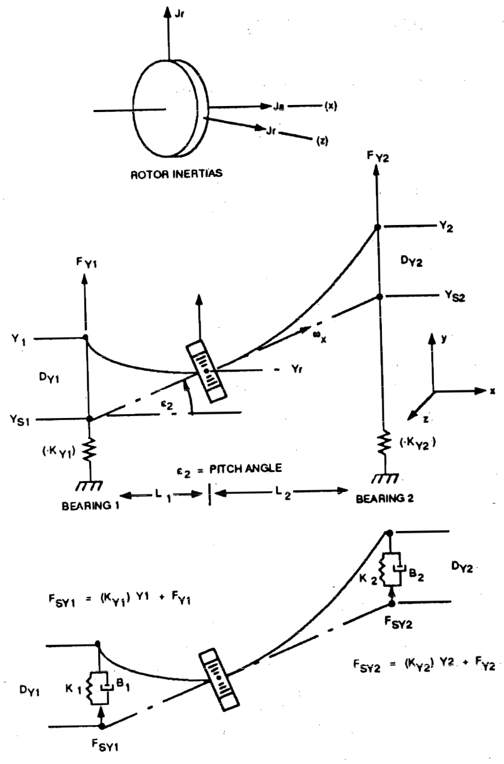

Figure 1. Side view of the Jeffcott Model showing references.

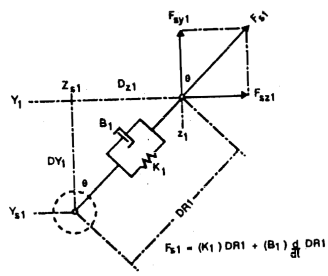

Figure 2. The spring-dashpot combination at the leftmost support.

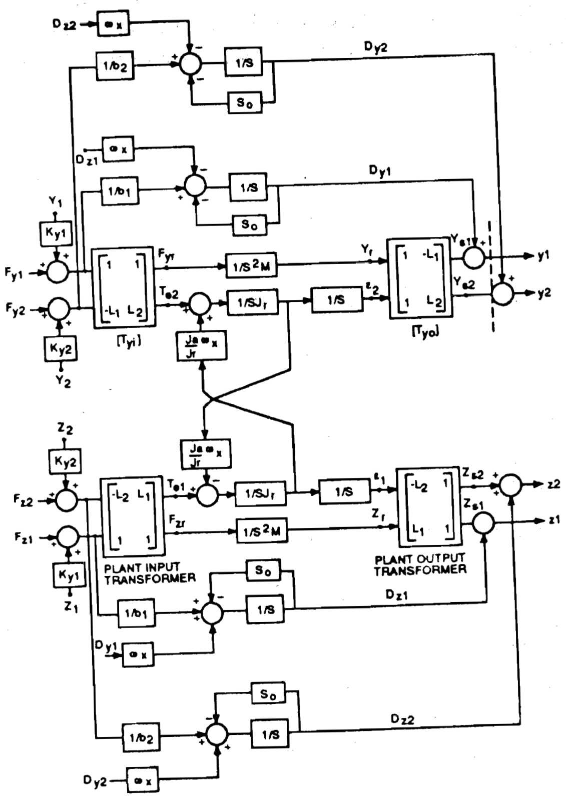

Figure 3. Block diagram of the Jeffcott rotor.