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Autobalancing of AMB Systems Using a Differential Regulator Based Output Regulation Approach

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Abstract

High speed rotating machines are subject to unbalance forces caused by residue weight. When the rotor's axis of geometry and its principal axis of inertia are not aligned, unbalance forces synchronous to the rotational speed cause the rotor to deflect from the geometric center and enter a whirling motion. To reduce the effects that the rotor unbalance has on high speed machineries supported by AMBs, the conventional approach has been to either generate counteracting bearing forces or to shift the rotating axis in such a way that the shaft is rotating force-free or performing auto-balancing. In this paper, a differential regulator based output regulation approach is presented to address the autobalancing of AMB systems for varying rotational speeds. After formulating the output regulation problem with a time-varying exosystem, it is observed that the compensator gains can be obtained based on the solution of the differential regulator equation (DRE) and the output regulation objective can be achieved approximately with a small bounded error in the regulated output. The proposed method is verified in simulation for autobalancing with both varying and constant rotational speeds on a flexible rotor AMB test rig.

Key words : AMB, auto-balancing, output regulation, time-varying exosystem, differential regulator equation

1. Introduction

High speed rotating machines are subject to unbalance forces caused by residue unbalance, which always exists in a newly machined rotor (Schweitzer and Maslen, 2009). When the rotor's axis of geometry and its principal axis of inertia are not aligned, rotor unbalance results in disturbance forces that are synchronous to the rotating speed and drive the rotor into whirling motion. Whirling causes the rotor to deflect from the geometric center and leads to potential instability. In a rotating machine supported by active magnetic bearings (AMBs), the unbalance problem can be addressed using the autobalancing technique, by which the rotor spinning axis coincides with the principal axis of inertia. As a result, the supporting force generated by AMBs can be significantly reduced. Many researchers have investigated autobalancing at a constant rotational speed and different strategies have been proposed (Mizuno and Higuchi, 1990, Yoon et al., 2016). When the rotational speed varies, the conventional approach is to switch the compensator gains that have been designed and saved in the memory. However, switching of regulator gains may result in bumpy transience.

In this paper, a differential regulator based output regulation approach is presented to addressing the autobalancing problem of AMB systems for varying rotational speeds. The problem of output regulation is to design a controller for disturbance rejection and/or reference tracking, while the disturbance or reference signal is generated by a known dynamic system called exosystem. After formulating the output regulation problem based on a time-varying exosystem, it is observed that the compensator gains can be obtained based on the solution of a differential regulator equation (DRE). Since AMB systems are of non-minimum phase, to ensure the boundedness of the compensator gains, the original normal form is reformulated and an unified gradient method is adopted to guarantee the residue error in the output regulation is minimized. Then the compensator gains are continuously generated to closely approach the output regulation objective with a small error in the regulated output. To apply output regulation to AMB systems for autobalancing, the unbalance force is modeled by the exosystem and the AMB force defines the error to be regulated (Yoon et al., 2016). When the rotational speed varies, the exosystem becomes time-varying, and the proposed differential regulator based output regulation approach is adopted to generate the desired bounded compensator gains that minimize the AMB control force to achieve autobalancing.

The paper is organized as follows. First the differential regulator based output regulation method is presented in Section 2. Section 3 describes the approach of obtaining bounded solutions to the DRE for non-minimum phase system. In Section 4 the autobalancing in AMB systems is described for varying rotational speeds, and the flexible rotor AMB test rig is briefly introduced. Simulation results are then shown in Section 5. Finally, Section 6 concludes the paper.

2. The Differential Regulator Based Output Regulation

The output regulation problem has been actively studied since it was formulated by (Francis, 1977) for linear systems and by (Isidori and Byrnes, 1990) for nonlinear systems. Recently, the output regulation approach has been applied to achieve disturbance rejection for systems with input delays (Yoon et al., 2016).

Consider a linear time-invariant plant in the following form

$$\dot{x}(t) = Ax(t) + Bu(t) + Pw(t), \tag{1a}$$

$$y(t) = Hx(t), \tag{1b}$$

$$e(t) = Cx(t) + Du(t) + Qw(t), \tag{1c}$$

where the state vector $x \in \mathbf{R}^n$, the control input vector $u \in \mathbf{R}^m$, the output vector $y \in \mathbf{R}^r$, the error to be regulated $e \in \mathbf{R}^r$.

Consider also a linear time-varying exosystem

$$\dot{w}(t) = S(t)w(t), \tag{2}$$

where $w \in \mathbf{R}^p$ represents the disturbance to be rejected ($Pw(t)$) and the reference signal to be tracked ($Q(w(t))$, $S(t)$ is bounded and smooth with bounded derivatives.

For convenience, a matrix $A(t)$ is said to be exponentially stable if the linear system $\dot{x} = A(t)x$ is exponentially stable, which means that the corresponding state transition matrix $\phi_A(t, t_0)$ satisfies $||\phi_A(t, t_0)|| \leq c_0 e^{-\alpha(t-t_0)}$ for $t \geq t_0$ for some $c_0 > 0$ and $\alpha > 0$.

The objective of the output regulation is to find a control law such that

- The closed-loop system is asymptotically stable when $w \equiv 0$;
- The error $e(t)$ converges to zero, for any initial conditions of the plant and the exosystem.

Two assumptions are made for the solvability of the output regulation problem.

Assumption 1. *The system* (1) *is detectable and stabilizable when* $w \equiv 0$*.*

Assumption 2. *The state transition matrix for the time-varying exosystem is uniformly bounded for all* t *and* t_0*, that is,* $c_1 \leq ||\phi_S(t, t_0)|| \leq c_2$ *for some positive constants* c_1 *and* c_2*.*

Assumption 1 is required for the existence of a control law that asymptotically stabilizes the system (1) when $w \equiv 0$. Assumption 2 does not affect the generality of the problem and keeps the output regulation problem nontrivial.

2.1. Output Regulation by State Feedback

Given that a stabilizing control law exists for the disturbance-free system, the following lemma provides a solution to the output regulation problem by state feedback.

Lemma 1. *Consider system* (1) *satisfying Assumptions 1-2, and define gain* κ *such that* $A + B\kappa$ *is Hurwitz. The state feedback control law*

$$u(t) = \kappa x(t) + (R(t) - \kappa\Pi(t))w(t), \tag{3}$$

achieves output regulation if and only if matrices $R(t) \in \mathbf{R}^{m\times a}$ *and* $\Pi(t) \in \mathbf{R}^{n\times a}$ *exist and satisfy the following differential regulator equation* (DRE)

$$\dot{\Pi}(t) + \Pi(t)S(t) = A\Pi(t) + BR(t) + P, \tag{4a}$$

$$\lim_{t\to\infty}(C\Pi(t) + DR(t) + Q) = 0. \tag{4b}$$

Proof. The proof starts with defining the state transformation $z(t) = x(t) - \Pi(t)w(t)$. The dynamics of $z(t)$ and the regulation error $e(t)$ can be written as

$$\dot{z}(t) = Az(t) + Bu(t) + A\Pi(t)w(t) + Pw(t) - \Pi(t)S(t)w(t) - \dot{\Pi}(t)w(t),$$
$$e(t) = Cz(t) + Du(t) + C\Pi(t)w(t) + Qw(t).$$

Consequently, the control law (3) can be written as $u(t) = \kappa z(t) + R(t)w(t)$. Let $R(t)$ and $\Pi(t)$ be solution of the DRE (4), the equations for $\dot{z}(t)$ and $e(t)$ simplify to

$$\dot{z}(t) = (A + B\kappa)z(t), \tag{5a}$$
$$e(t) = (C + D\kappa)z(t). \tag{5b}$$

Since $A + B\kappa$ is Hurwitz, we have $\lim_{t\to\infty} z(t) = 0$ and hence $\lim_{t\to\infty} e(t) = 0$. Thus, control law (3) solves the output regulation problem.

To demonstrate the necessary condition, assuming the state feedback control law (3) is a solution of the output regulation problem, then $e(t)$ become

$$e(t) = (C + D\kappa)z(t) + (C\Pi(t) + DR(t) + Q)w(t),$$

which must satisfy that $\lim_{t\to\infty} e(t) = 0$. Since Assumption 2 states that $w(t)$ is nonvanishing, and $e(t)$ approaches 0 for arbitrary initial conditions if $\lim_{t\to\infty} z(t) = 0$ and (4b) is true. Furthermore, it was assumed that under $u(t) = \kappa z(t)+R(t)w(t)$, (5) are established for $z(t)$ and $e(t)$ to be asymptotically stable. For $\lim_{t\to\infty} z(t) = 0$ to be true under Assumption 2, it requires (4a) to be satisfied.

2.2. Output Regulation by Output Feedback

Consider the combined perturbed system with $\chi = [x^{\mathrm{T}}\ w^{\mathrm{T}}]^{\mathrm{T}}$, and

$$\dot{\chi}(t) = \begin{bmatrix} A & P \\ 0 & S(t) \end{bmatrix} \chi(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \tag{6a}$$
$$y(t) = \begin{bmatrix} H & 0 \end{bmatrix} \chi(t), \tag{6b}$$
$$e(t) = \begin{bmatrix} C & Q \end{bmatrix} \chi(t) + Du(t). \tag{6c}$$

Lemma 2. *Consider the system* (6) *satisfying Assumptions 1-2, and let the control law*

$$u(t) = \kappa x(t) + (R(t) - \kappa\Pi(t))w(t),$$

be the solution to the state feedback output regulation problem from Lemma 1. The output feedback control law

$$\dot{\hat{\chi}}(t) = \mathbb{A}\hat{\chi}(t) + \mathbb{B}u(t) + L(t)(\mathbb{C}\hat{\chi}(t) - y(t)), \tag{7a}$$
$$u(t) = \begin{bmatrix} \kappa & R(t) - \kappa\Pi(t) \end{bmatrix} \hat{\chi}(t), \tag{7b}$$

where

$$\hat{\chi} = \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix},\ \mathbb{A}(t) = \begin{bmatrix} A & P \\ 0 & S(t) \end{bmatrix},\ \mathbb{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},\ \mathbb{C} = \begin{bmatrix} H & 0 \end{bmatrix},$$

$\hat{x}$ and $\hat{w}$ are the estimated states, achieves the output regulation objectives if the transition matrix ϕ_{Ξ^L} corresponding to $\Xi^L(t) = \mathbb{A}(t) + L(t)\mathbb{C}$ and $L \in \mathbf{R}^{r\times m}$ satisfies $\|\phi_{\Xi^L}(t, t_0)\| \le k_L e^{-l_0(t-t_0)}$ for some $k_L > 0$ and $l_0 > 0$.

Proof. The proof starts with defining the state transformation $z(t) = x(t) - \Pi(t)w(t)$. Let $\hat{z} = \hat{x} - \Pi\hat{w}$ and $\tilde{z} = \hat{z} - z$. The closed-loop equation of the $z(t)$ and the regulation error $e(t)$ can be written as

$$\dot{z}(t) = Az(t) + A\Pi(t)w(t) + Pw(t) - \Pi(t)S(t)w(t) - \dot{\Pi}(t)w(t) + B\kappa\hat{z}(t) + BR(t)\hat{w}(t),$$
$$e(t) = Cz(t) + C\Pi(t)w(t) + Qw(t) + D\kappa\hat{z}(t) + DR(t)\hat{w}(t).$$

After expanding $\hat{z} = \tilde{z} + z$, $\hat{w} = \tilde{w} + w$ and following $\Pi(t)$ and $R(t)$ as solutions of the DRE (4)

$$\dot{z}(t) = (A + B\kappa)z(t) + \begin{bmatrix} B\kappa & B(R(t) - \kappa\Pi(t)) \end{bmatrix} \tilde{\chi}(t),$$
$$e(t) = (C + D\kappa)z(t) + \begin{bmatrix} D\kappa & D(R(t) - \kappa\Pi(t)) \end{bmatrix} \tilde{\chi}(t),$$

and the following equation can be obtained for the state estimation errors $\tilde{\chi}(t) = \hat{\chi}(t) - \chi(t)$

$$\dot{\tilde{\chi}}(t) = (A(t) + L(t)C)\tilde{\chi}(t). \tag{8}$$

With the assumption that $\|\phi_{\Xi^L}(t, t_0)\| \leq k_L e^{-l_0(t-t_0)}$ for $\Xi^L(t) = \mathbb{A}(t) + L(t)\mathbb{C}$, the system (8) is asymptotically stable. Thus, both the states $z(t)$ and state estimation errors $\tilde{\chi}(t)$ asymptotically approach zero, $\lim_{t\to\infty} e(t) = 0$.

3. Bounded Regulator Gains for Non-minimum Phase Systems

In the previous section, the solution to the output regulation problem with time-varying exosystem dynamics was presented in terms of the DRE in (4). For minimum phase systems, the solution to (4) is always bounded, and numerical methods can be used to construct the output regulation control law. When the system is of non-minimum phase, such as AMB systems, the solution to (4) may become unbounded and the applicability of the current approach is restricted to a small number of special cases (Shim et al., 2010). In this section we propose a sub-optimal solution to the output regulation problem for non-minimum phase systems, such that the solution to the corresponding DRE is always bounded.

A coordinate transformation is adopted to convert the system (1) into a normal form as described in (Isidori, 1995). Denote the jth row of the matrix C by C_j. For a set of positive integers $r_1, r_2, \cdots, r_p$, where p is the number of outputs, the row vector τ^i_j is defined as $\tau^i_1 = C_i$, $1 \leq i \leq p$, and $\tau^i_{j+1} = \tau^i_j A$, $1 \leq j \leq r_i - 1$. Also define the set of row vectors $\check{\tau}^i_1 = Q_i$, $1 \leq i \leq p$, and $\check{\tau}^i_{j+1} = \check{\tau}^i_j S + \tau^i_j P$, $1 \leq j \leq r_i - 1$.

Assumption 3. *There exist positive integers $r_1, r_2, \ldots, r_p$ known as the 'vector relative degree' (Isidori et al., 2003) such that $r = r_1 + r_2 + \cdots + r_p \leq n$, $\tau^i_j B = 0$ for $1 \leq i \leq p$ and $1 \leq j \leq r_i - 1$, and the rank of the matrix $\Psi = \begin{bmatrix} (\tau^1_{r_1})^{\mathrm{T}} & (\tau^2_{r_2})^{\mathrm{T}} & \cdots & (\tau^p_{r_p})^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} B \in \mathbf{R}^{p\times m}$ is p.*

Define the matrices $\bar{T}^x = \begin{bmatrix} (\tau^0_1)^{\mathrm{T}} & \cdots & (\tau^0_{n-r})^{\mathrm{T}} & (\tau^1_1)^{\mathrm{T}} & \cdots & (\tau^1_{r_1})^{\mathrm{T}} & (\tau^2_1)^{\mathrm{T}} & \cdots & (\tau^p_{r_p})^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, and $T^w = \begin{bmatrix} 0 & \cdots & 0 & (\check{\tau}^1_1)^{\mathrm{T}} & \cdots & (\check{\tau}^1_{r_1})^{\mathrm{T}} & (\check{\tau}^2_1)^{\mathrm{T}} & \cdots & (\check{\tau}^p_{r_p})^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, where $\tau^0_1, \tau^0_2, \ldots, \tau^0_{n-r}$ are row vectors of appropriate dimensions. In (Isidori et al., 2003), it shows that if Assumption 3 is true, then there exists a w-dependent coordinate transformation, $\begin{bmatrix} \bar{\beta} \\ \zeta \end{bmatrix} = \bar{T}^x x + T^w w$, such that system (1) takes the following normal form

$$\begin{aligned}
&\dot{\bar{\beta}} = \bar{\Gamma}\bar{\beta} + \bar{\Lambda}\zeta + \bar{\Theta}w,\\
&\dot{\zeta}^j_{i-1} = \zeta^j_i, \text{ for } i = 2, 3, \cdots, r_j \text{ and } j = 1, 2, \cdots, p,\\
&[\dot{\zeta}^1_{r_1}, \ldots, \dot{\zeta}^p_{r_p}]^{\mathrm{T}} = \bar{L}\bar{\beta} + M\zeta + \Psi u + Nw,\\
&e = [\zeta^1_1, \ldots, \zeta^p_1]^{\mathrm{T}}.
\end{aligned}$$

Assumption 4. *There exists a coordinate transformation T^β such that, under the coordinate change $\beta = \begin{bmatrix} \beta^a \\ \beta^s \end{bmatrix} = T^\beta \bar{\beta}$, $\beta^a \in \mathbf{R}^k$, $\beta^s \in \mathbf{R}^{n-r-k}$, the zero dynamics $\dot{\bar{\beta}} = \bar{\Gamma}\bar{\beta}$ becomes*

$$\begin{bmatrix} \dot{\beta}^a \\ \dot{\beta}^s \end{bmatrix} = \begin{bmatrix} \Gamma^a & 0 \\ 0 & \Gamma^s \end{bmatrix} \begin{bmatrix} \beta^a \\ \beta^s \end{bmatrix} = \Gamma\beta, \tag{9}$$

where Γ^a and Γ^s are exponentially anti-stable and stable, respectively.

Based on Assumptions 3 and 4, the coordinate transformation $\begin{bmatrix} \beta \\ \zeta \end{bmatrix} = \begin{bmatrix} T^\beta & 0 \\ 0 & I_{r\times r} \end{bmatrix} \bar{T}^x x + T^w w = T^x x + T^w w$, yields the following normal form for system (1)

$$\dot{\beta} = \Gamma\beta + \Lambda\zeta + \Theta w, \tag{10a}$$
$$\dot{\zeta}^j_{i-1} = \zeta^j_i, \text{ for } i = 2, 3, \cdots, r_j \text{ and } j = 1, 2, ..., p, \tag{10b}$$
$$[\dot{\zeta}^1_{r_1}, \ldots, \dot{\zeta}^p_{r_p}]^{\mathrm{T}} = L\beta + M\zeta + \Psi u + Nw, \tag{10c}$$
$$e = [\zeta^1_1, \ldots, \zeta^p_1]^{\mathrm{T}}. \tag{10d}$$

For a system with the above normal form, an explicit solution to the DRE is found in (Shim et al., 2010).

Theorem 1 (Shim et al., 2010). *Under Assumptions 1-4, the linear differential matrix equation*

$$\dot{\Pi}^\beta(t) + \Pi^\beta(t)S(t) = \Gamma\Pi^\beta(t) + \Theta \tag{11}$$

always have a solution $\Pi^{\beta}(t) = [(\Pi^{\beta a}(t))^{\mathrm{T}} \quad (\Pi^{\beta s}(t))^{\mathrm{T}}]^{\mathrm{T}}$ *and they can be obtained as follows,*

$$\Pi^{\beta a}(t) = -\int_t^{\infty} \Phi_{\Gamma^a}(t,\sigma)\Theta^a\Phi_s(\sigma,t)d\sigma,$$

$$\Pi^{\beta s}(t) = \Phi_{\Gamma^s}(t,t_0)\Pi_0^{\beta s}\Phi_s(t_0,t) + \int_{t_0}^{t} \Phi_{\Gamma^s}(t,\sigma)\Theta^s\Phi_s(\sigma,t)d\sigma,$$

where Θ^a *and* Θ^s *are upper* k *rows and the lower* $n-k-r$ *rows of* Θ*, respectively. The matrix* $\Pi_0^{\beta s}$ *is any constant* $(n-r-k)\times q$ *matrix. The DRE* (4) *has a solution given as*

$$\Pi = (T^x)^{-1}\left(\begin{bmatrix}\Pi^{\beta}\\ 0_{r\times q}\end{bmatrix} - T^w\right), \tag{12a}$$

$$R = -(\Psi)^+(L\Pi^{\beta} + N), \tag{12b}$$

where $(\Psi)^+$ *is the right-inverse of* Ψ.

Note that the solution to the DRE for the stable zero dynamics $\Pi^{\beta s}$ can be found by calculating the corresponding definite integral with finite limits in Theorem 1. On the other hand, the solution to the anti-stable part of the zero dynamics $\Pi^{\beta a}$ is more difficult to obtain, and the infinite upper limit of the closed-form solution can lead to numerical error during the implementation. In the remainder of this section, we present an alternative approach to Theorem 1, and we develop a numerically robust suboptimal solution to the output regulation problem.

In order to guarantee a bounded solution to the DRE, the unstable zero dynamics of (10) is first stabilized. Assuming a gain K_r exists such that all the eigenvalues of $(\Gamma + \Lambda K_r)$ are exponentially stable, we rewrite (10a) as

$$\dot{\beta} = (\Gamma + \Lambda K_r)\beta + \Lambda(\zeta - K_r\beta) + \Theta w.$$

Let $\tilde{\zeta} = \zeta - K_r\beta$ and Equation (10) then becomes

$$\dot{\beta} = (\Gamma + \Lambda K_r)\beta + \Lambda\tilde{\zeta} + \Theta w, \tag{13a}$$

$$\dot{\tilde{\zeta}}_{i-1}^{j} = \tilde{\zeta}_i^{j}, \text{ for } i = 2,3,\cdots,r_j \text{ and } j = 1,2,...,p, \tag{13b}$$

$$\begin{aligned}[\dot{\tilde{\zeta}}_{r_1}^{1},\ldots,\dot{\tilde{\zeta}}_{r_p}^{p}]^{\mathrm{T}} &= L\beta + M\tilde{\zeta} + \Psi u + Nw - K_r(\Gamma\beta + \Lambda(\tilde{\zeta} + K_r\beta) + \Theta w),\\ &= (L - K_r\Gamma + MK_r - K_r\Lambda K_r)\beta + (M - K_r\Lambda)\tilde{\zeta} + \Psi u + (N - K_r\Theta)w,\end{aligned} \tag{13c}$$

$$e = [\tilde{\zeta}_1^{1},\ldots,\tilde{\zeta}_1^{p}]^{\mathrm{T}} + K_r\beta. \tag{13d}$$

Based on the updated normal form (13), let the control input $u(t)$ drive the states $\tilde{\zeta}$ to zero. The regulated error then becomes $e = K_r\beta$. In addition, for the zero dynamics in (13a), the solution to the corresponding DRE

$$\dot{\Pi}^{\beta}(t) = (\Gamma + \Lambda K_r)\Pi^{\beta}(t) - \Pi^{\beta}(t)S(t) + \Theta,$$

is always bounded. As $\tilde{\zeta} \to 0$, The system equation (13) reduces to

$$\dot{\beta} = (\Gamma + \Lambda K_r)\beta + \Theta w, \tag{14a}$$

$$e = K_r\beta. \tag{14b}$$

To regulate the residual error in (14b), we minimize the H_2 norm of the transfer function from w to e, given by $G_e(s) = K_r(sI - \Gamma - \Lambda K_r)^{-1}\Theta$.

The unified gradient method from (Hu et al., 2004) is utilized to minimize the H_2 norm of (14). Assuming that $\Gamma + \Lambda K_r$ is stable, the H_2 norm of $G_e(s)$ is given by

$$||G_e(s)||_2^2 = tr(\Theta^{\mathrm{T}}P\Theta) =: J_1 \tag{15}$$

where $P = P^{\mathrm{T}} \geq 0$ satisfies the Lyapunov equation

$$(\Gamma + \Lambda K_r)^{\mathrm{T}}P + P(\Gamma + \Lambda K_r) = -K_r^{\mathrm{T}}K_r. \tag{16}$$

In the case where the poles of $\Gamma + \Lambda K_r$ are not fixed *a priori*, the gain K_r may become unbounded to make the H_2 norm in (15) small. Instead, the following H_2 optimization problem under pole assignment constraints is considered

$$\inf_{K_r} J_1(K_r), \tag{17}$$

$$\text{s.t. } V^{-1}(\Gamma + \Lambda K_r)V = \Lambda_r,$$

where Λ_r is a real block-diagonal matrix with its eigenvalues corresponding to the desired pole locations, V is a nonsingular matrix satisfying $\Gamma V - V\Lambda_r = -\Lambda U$ and $K_r = UV^{-1}$ for a free parameter U.

For the constrained optimization (17), it was shown in (Hu et al., 2004) that

$$\frac{\partial J_1}{\partial K_r} = 2\Lambda^{\mathrm{T}} PX, \tag{18}$$

where X is the unique solution to the Lyapunov equation $(\Gamma + \Lambda K_r)X + X(\Gamma + \Lambda K_r)^{\mathrm{T}} = -\Theta\Theta^{\mathrm{T}}$. Furthermore, the gradient of the objective function about the free parameter matrix U was given by

$$\frac{\partial J_1}{\partial U} = \frac{\partial J_1}{\partial K_r}(V^{-1})^{\mathrm{T}} + \Lambda^{\mathrm{T}} Y^{\mathrm{T}}, \tag{19}$$

where $\partial J_1/\partial K_r$ is given in (18), and Y is the unique solution to the Sylvester equation

$$Y\Gamma - \Lambda_r Y = V^{-1}\left(\frac{\partial J_1}{\partial K_r}\right)^{\mathrm{T}} K_r. \tag{20}$$

A local minimum to (17) can then be found numerically by evaluating the gradient (19) about the free variable U.

4. Autobalancing and Test Rig

In this section we described the autobalancing method for AMB systems, and we introduce the AMB test rig that we use to verify our results.

4.1. Autobalancing in an AMB System

We consider an active magnetic bearing (AMB) system operating below the first critical speed of the rotor. The lateral dynamics of the AMB system are described the by following equation,

$$\begin{bmatrix}\dot\zeta_x(t)\\ \dot\zeta_y(t)\end{bmatrix} = \begin{bmatrix}A_\zeta & \omega G_r\\ -\omega G_r & A_\zeta\end{bmatrix}\begin{bmatrix}\zeta_x(t)\\ \zeta_y(t)\end{bmatrix} + \begin{bmatrix}B_c & 0\\ 0 & B_c\end{bmatrix}\begin{bmatrix}i_{c,x}(t)\\ i_{c,y}(t)\end{bmatrix} + \sum_{j=1}^{s}\begin{bmatrix}B_{d,j} & 0\\ 0 & B_{d,j}\end{bmatrix}\begin{bmatrix}w_{x,j}(t)\\ w_{y,j}(t)\end{bmatrix},$$

where $\zeta = \begin{bmatrix}\zeta_x\\ \zeta_y\end{bmatrix}$ describes the states of the AMB system lateral dynamics, ω is the rotating speed, G_r is the gyroscopic matrix, $i_c = \begin{bmatrix}i_{c,x}\\ i_{c,y}\end{bmatrix}$ is the control current vector of the AMB actuators operating in the current mode, and $w = \begin{bmatrix}w_{x,j}\\ w_{y,j}\end{bmatrix}$ represent the disturbance force generated by the j^{th} unbalance mass on the rotor. The unbalance mass creates disturbances forces that are synchronous to the shaft rotating speed

$$w_{x,j} = \epsilon_j\omega^2\cos(\omega t + \theta_j), \tag{21a}$$

$$w_{y,j} = \epsilon_j\omega^2\sin(\omega t + \theta_j), \tag{21b}$$

where ϵ_j is the unbalance eccentricity and θ_j is the phase angle. The disturbance forces can be represented as the outputs of the following time-varying exosystem,

$$\begin{bmatrix}\dot w_{x,j}\\ \dot w_{y,j}\end{bmatrix} = \begin{bmatrix}0 & -\omega(t)\\ \omega(t) & 0\end{bmatrix}\begin{bmatrix}w_{x,j}\\ w_{y,j}\end{bmatrix}. \tag{22}$$

An autobalancing method for AMB systems minimizes the magnitude of the disturbance forces $w_{x,j}$ and $w_{y,j}$ by aligning the rotor's center of mass with its axis of rotation. This can be formulated as an output regulation problem by defining the regulated error signal e_f to be the applied AMB forces f_{amb}

$$e_f = f_{amb} = K_x C_{amb}\zeta + K_i i_c,$$

where K_x and K_i are the AMB open loop stiffness and the AMB current gain, respectively. The combined AMB system and exosystem equations for the output regulation problem then becomes

$$\dot\zeta(t) = \mathbf{A}\zeta(t) + \mathbf{B}i_c(t) + \sum_{j=1}^{s}\mathbf{P}_j w_j(t), \tag{23a}$$

$$\dot w_j(t) = \mathbf{S}(t)w_j(t),\ \text{for } j = 1, 2, \cdots, s, \tag{23b}$$

$$e_f(t) = K_x C_{amb}\zeta(t) + K_i i_c(t). \tag{23c}$$

The controller applied in the simulation takes the state feedback form $i_c(t) = \kappa\zeta(t) + (R(t) - \kappa\Pi(t))w(t)$, where κ is designed using the LQR approach, $R(t)$ and $\Pi(t)$ are generated based on Eq. (12). A K_r is determined through the unified gradient method to regulate the residual error in $e_f(t)$.

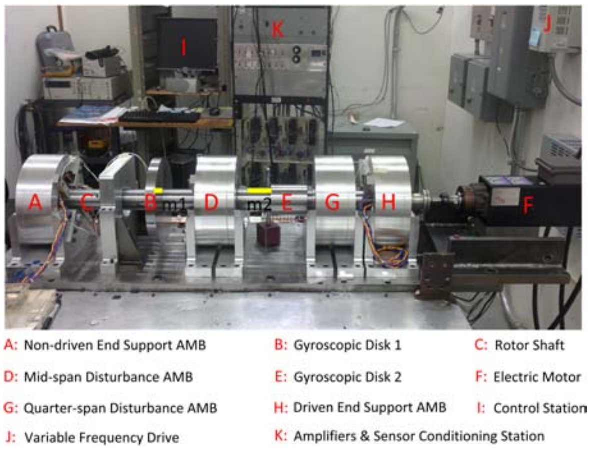

Fig. 1 The flexible rotor AMB test rig.

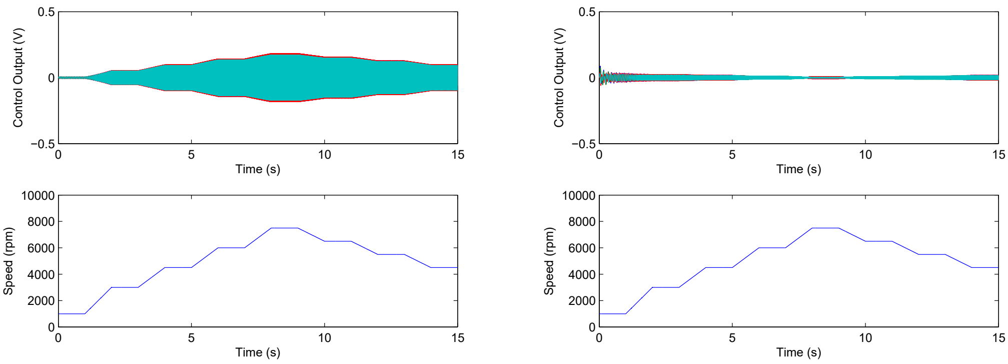

Fig. 2 Simulated control voltages without/with differential regulator as the rotating speed varies between 1,000 and 7,500 rpm using the state feedback.

4.2. The Flexible Rotor AMB Test Rig

The ROMAC flexible rotor AMB test rig is adopted to verify the proposed autobalancing method. The flexible rotor AMB test rig is a research platform constructed in the ROMAC laboratory. The original purpose of this test rig was to emulate an industrial size centrifugal gas compressor and perform advanced control designs (Di and Lin, 2014). In particular, Disk 1 and Disk 2 emulate the wheels in a compressor. There are four AMBs in the test rig. Two radial support AMBs are located at the non-driven end (NDE) and driven end (DE) of the rotor. One exciter AMB is at the mid span and the other is at the quarter span of the rotor. This combination of four radial AMBs allows the simulation of different operating conditions of a compressor. Shown in Fig. 1 illustrates the assembled AMB test rig.

5. Simulation Results

The proposed method is verified by simulation for autobalancing at both varying rotational speeds and constant speeds on the flexible rotor AMB test rig model. Since AMB systems are of non-minimum phase, a gain K_r determined based on the unified gradient method is applied to stabilize the unbounded compensator gains and to ensure $\inf_{K_r} J_1(K_r)$ is achieved. Shown in Fig. 2 are the control voltages of the four radial axes as the rotating speed varies between 1,000 and 7,500 rpm using the state feedback, without and with differential regulator, respectively. It can be observed that the control voltage is significantly reduced over the wide speed range. Fig. 3 shows the simulated control voltages and rotor displacements without and with the differential regulator at 7,500 rpm using the state feedback. The vibration levels under both cases are similar while the control voltage is significantly reduced with the differential regulator.

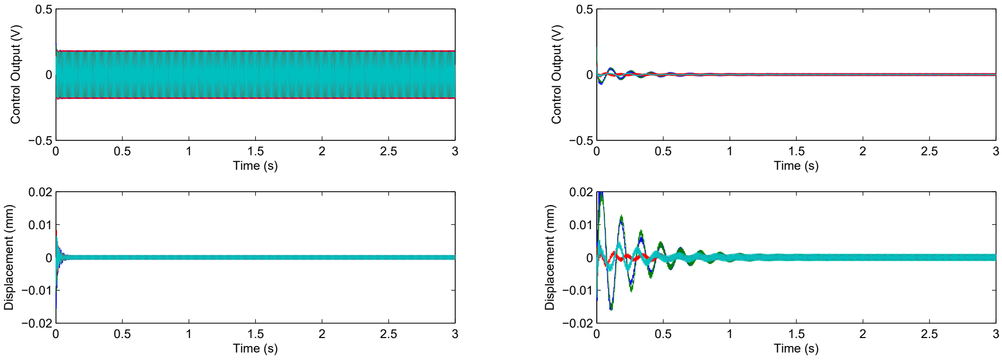

Fig. 3 Simulated control voltages and rotor displacements without/with differential regulator at 7,500 rpm using the state feedback.

6. Conclusions

This paper presents a differential regulator based output regulation approach to address the autobalancing of AMB systems for varying rotational speeds. The output regulation problem with time-varying exosystem can be addressed from the solution of a differential regulator equation (DRE). By applying the proposed method, the compensator gains can be continuously updated to closely approach the output regulation objective with a small bounded error in the regulated output. Both state and output feedback designs have been developed. When the rotational speed changes for an AMB system, the unbalance force based exosystem becomes time-varying, and the proposed differential regulator has been applied to regulate the AMB control force to achieve autobalancing. The simulation results have verified the effectiveness of the method under time-varying rotational speeds and constant speed.

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