Nitsche-type Mortaring for Simulation of Electrical Machines

Armin Fohler[a], Walter Zulehner[b]

[a] Linz Center of Mechatronics, Altenberger Straße 69, 4040 Linz, Austria, armin.fohler@lcm.at
[b] Johannes Kepler University Linz, Altenberger Straße 69, 4040 Linz, Austria

***Abstract*—We will present a finite element program implementing the idea of Nitsche-type mortaring in the framework of simulating electrical machines. Using this method it is possible to split the domain into subdomains only connected via a function on the interface. From this approach we gain the advantage that adjacent meshes (e.g. rotor and air gap) do not have to fit together nodewise. The mortaring will be used on the interface between rotor and air gap as well as on the symmetry edges of the motor, since only a part of the machine is considered in the simulation due to motor symmetry.**

I. Introduction

The program at issue was designed to deal with the magnetostatic case of Maxwell's equations in 2D, as it occurs as a simplification in the simulation of electrical machines. Starting with:

$$\begin{aligned}\operatorname{curl}\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t};\\ \operatorname{curl}\mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t};\\ \operatorname{div}\mathbf{D} &= \rho;\\ \operatorname{div}\mathbf{B} &= 0;\end{aligned}$$

with material laws:

$$\begin{aligned}\mathbf{B} &= \mu(|\mathbf{H}|)\mathbf{H} + \mathbf{M};\\ \mathbf{D} &= \epsilon\mathbf{E} + \mathbf{P};\\ \mathbf{J} &= \sigma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right).\end{aligned}$$

we can derive the magnetostatic case. Here $\mathbf{E}$ denotes the electric field intensity, $\mathbf{P}$ the polarization. $\frac{1}{\mu} = \nu$, with μ the permeability and ν the reluctivity, ρ stands for the charge density. The electric conductivity is represented by σ, v the velocity and ϵ is the permitivity. Under the assumptions, that the magnetic flux density $\mathbf{B}$, the magnetic field intensity $\mathbf{H}$ and the electric current density $\mathbf{J}$ are time-independent and $\frac{\partial \mathbf{D}}{\partial t} \ll \mathbf{J}$, with $\mathbf{D}$ denoting the electric flux density we arrive at the magnetostatic formulation.
Those assumptions are reasonable in the framework of electrical machines since we are dealing with slowly varying magnetic fields.
On a single computational domain with $\partial\Omega$ symbolizing its boundary we get the following boundary value problem:

Find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{aligned}-\operatorname{div}\left(\nu\left(|\nabla u|\right)\nabla u\right) &= J_3 - \frac{\partial M_1}{\partial x_2} + \frac{\partial M_2}{\partial x_1} && \text{in } \Omega\\ u &= 0 && \text{on } \partial\Omega\end{aligned}$$

If we consider a computational domain divided into subdomains Ω_i with interfaces Γ_{ij} between the domains Ω_i and Ω_j we have to impose additional conditions on the interfaces. In our case those subdomains will be the rotor, the stator and the air gap. The adapted boundary value problem is he following:

Find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{aligned}-\operatorname{div}\left(\nu\left(|\nabla u_i|\right)\nabla u_i\right) &= J_{3,i} - \frac{\partial M_{1,i}}{\partial x_2} + \frac{\partial M_{2,i}}{\partial x_1} && \text{in } \Omega_i\\ u_i &= u_j && \text{on } \Gamma_{ij}\\ \nu\left(|\nabla u_i|\right)\nabla u_i \cdot n_i - M_i\cdot\tau_i &&&\\ = \nu\left(|\nabla u_j|\right)\nabla u_j \cdot n_j - M_j\cdot\tau_j &&& \text{on } \Gamma_{ij}\\ u &= 0 && \text{on } \partial\Omega\end{aligned}$$

In this system of equations M denotes the magnetization if the domain is a permanent magnet, J_3 is the only nonzero component of the current density vector and n_i stands for the outer normal vector and τ_i for the tangential vector of the subdomain Ω_i. The reluctivity ν depends nonlinearily on u.
The second condition states that the solution u does not have jumps on the interface between two domains. The third condition characterizes the flux in normal direction.

II. Nonlinear materials

During computation we deal with the dependency of ν on u in nonlinear materials by applying a Newton method. For the termination criterion we measure the L^2-norm of the residual

$$F(.) := b - A\left(\nu\left(|\nabla u_{n-1}|\right)\right)u_n.$$

Where b denotes the right-hand-side of the multidomain formulation and A the matrix representing the linearized system. The index $n, n-1$ refers to the solution of the Newton method in the n and $n-1$ step respectively. If the residual of the last computed u_n for the current rotor-stator-constellation is smaller than ϵ times the residual of the initial guess u_{00} in the L^2-norm:

$$F(u_n) \leq \epsilon F(u_{00})$$

we accept the last computed u_n as solution. Here the tolerance parameter ϵ is chosen 10^{-6}.

III. Domain decomposition techniques

We want to approximate a full rotation of the motor without remeshing the whole domain in every discrete step. One possibility to do this is the moving band technique [1]. The idea is to fix the meshes in two adjacent domains and only generate a new mesh in a small region between those. If we can impose an equidistant mesh on the two adjacent parts we can even use the same mesh for each calculation. Using this approach we have a conforming mesh in every rotation step. The downside of this technique is that we cannot use nonconforming meshes and – as mentioned in [2] – in the 3D case the remeshing for each calculation is impractical.
Focusing on domain decomposition techniques which allow us to generate one mesh for each subdomain independently for all rotational displacements we find three major approaches:

- the mortar element method
- the discontinuous Galerkin method
- the Nitsche-type mortaring.

In the framework of the mortar element methods on the interface between two domains one domain is defined as the masterside and one as the slave. To glue both domains together Lagrange-multipliers are used. In the standard form this leads to a saddle point problem. By choosing special shape functions [3] for the Lagrange multipliers this saddle point form can be avoided [2].
The classical discontinuous Galerkin approach is formulated for each triangle of the mesh, see also [4]. The ansatz functions are discontinuous over the element interfaces. Stability is guaranteed by adding penalisation terms for the jumps of the solution between to adjacent triangles in the bilinear form. This can be easily generalized to the domain level.

IV. Nitsche-type mortaring

Originally the idea of Nitsche was to incorporate Dirichlet boundary data directly into the variational formulation and not – as usual – into the function space [5].
The same principle can be applied on an interface between two subdomains. In the literature we can find slightly different approaches all named Nitsche(-type) mortaring (see [6], [7]), we will follow the formulation proposed in [8]. The idea is similar to the incorporation of boundary conditions: The function which was given in the case of incorporating Dirichlet boundary condition is now an unknown function λ on the interface between the two domains. This unknown function represents the solution on the interface $u\big|_{\Gamma}$.
To get a continuous solution on the whole domain we have to ensure that there are no jumps between the solutions on adjacent subdomains. Here the function on the interface comes into play as we enforce the solution on each of these subdomains to match λ.
To show the difference between the standard variational formulation (1) in comparison to Nitsche-type mortaring (2) both are given below. The variational form of the boundary value problem for the single computational domain writes as:

Find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\int_\Omega \left(\nu\left(|\nabla u|\right)\nabla u - (M_1, M_2)_i^T\right)\cdot\nabla v dx = \int_\Omega J_3 v dx \quad (1)$$

$\forall v : \overline{\Omega} \to \mathbb{R}, v\big|_{\partial\Omega} = 0;$

whereas the Nitsche-type formulation based on the multidomain approach with the notation $u_i := u\big|_{\Omega_i}$ is given by:

Find $u : \overline{\Omega} \to \mathbb{R}$, $\lambda_{ij} : \Gamma_{ij} \to \mathbb{R}$ such that

$$\begin{aligned}
&\sum_i \Bigg(\int_{\Omega_i} \left(\nu\left(|\nabla u_i|\right)\nabla u_i - (M_1, M_2)_i^T\right)\cdot\nabla v_i dx \\
&\quad - \sum_j \underbrace{\int_{\Gamma_{ij}} \left(\nu\left(|\nabla u_i|\right)\nabla u_i \cdot n_i - (M_1, M_2)_i^T \cdot \tau_i\right)\left(v_i - \phi_{ij}\right) dS}_{(i)} \\
&\quad + \beta \sum_j \underbrace{\int_{\Gamma_{ij}} \left(\nu\left(|\nabla u_i|\right)\nabla v_i \cdot n_i\right)\left(u_i - \lambda_{ij}\right) dS}_{(ii)} \\
&\quad + \alpha \sum_j \underbrace{\int_{\Gamma_{ij}} \nu\left(|\nabla u_i|\right)\left(u_i - \lambda_{ij}\right)\left(v_i - \phi_{ij}\right) dS}_{(iii)} \Bigg) \\
&= \sum_i \int_{\Omega_i} J_3 \; v_i dx
\end{aligned} \quad (2)$$

$\forall v : \overline{\Omega} \to \mathbb{R}, v\big|_{\partial\Omega} = 0$ and $\forall \phi_{ij} : \Gamma_{ij} \to \mathbb{R}$

The functions ϕ appearing in the formula are the test functions on the boundary.
This formulation with interior interfaces Γ_{ij} can be easily adjusted to the case of symmetry edges.
We observe, that if the exact solution $\overline{u}$ of (1) is smooth enough, then it also solves the formulation of the Nitsche-type mortaring (2). This holds since the term (i) vanishes due to $\frac{\partial \overline{u}_i}{\partial n_i}\big|_{\Gamma_{ij}} = -\frac{\partial \overline{u}_j}{\partial n_j}\big|_{\Gamma_{ij}}$ on the interface between the domain Ω_i and Ω_j.
The terms (ii) as well as (iii) are equal to zero in case of such a solution $\overline{u}$ because $\overline{u}_i - \lambda_{ij} = 0$, considering $\overline{u}_i = \overline{u}_j = \lambda_{ij}$ on Γ_{ij}.
Therefore the formulation (2) is consistent.
Due to the fact that we solve the problem on a mesh we only get a discrete approximation of $\overline{u}$. Therefore the term (iii) will act as a penalisation term.
Via the weight factors α and β we can control the properties of the system. For example by assigning $\beta = -1$ we get a symmetric system. To ensure solvability we have to assign α in a proper way. For instance if $\beta = -1$, α has to be suitably large, whereas in the case $\beta = 1$ there are no restrictions for α.

The right hand side of the resulting system of linear equations can be represented by a matrix of the following form:

$$\begin{pmatrix} A_{\Omega_1} & & \alpha\tilde{C}_{\Gamma_1} + \beta B_{\Gamma_1} \\ & A_{\Omega_2} & \alpha\tilde{C}_{\Gamma_2} + \beta B_{\Gamma_2} \\ \alpha\tilde{C}_{\Gamma_1} - B_{\Gamma_1} & \alpha\tilde{C}_{\Gamma_2} - B_{\Gamma_2} & \alpha\hat{C}_{\Gamma_1} + \alpha\hat{C}_{\Gamma_2} \end{pmatrix} \begin{pmatrix} u_{\Omega_1} \\ u_{\Omega_2} \\ \lambda \end{pmatrix}$$

And the loadvector has the form:

$$\begin{pmatrix} \bar{c}_1(M_1, M_2)^T_{\Omega_1} + \bar{c}_2 J_{\Omega_1,3} - \bar{c}_3(M_1, M_2)^T_{\Omega_1} \cdot \tau_{\Omega_1} \\ \hat{c}_1(M_1, M_2)^T_{\Omega_2} + \hat{c}_2 J_{\Omega_2,3} - \hat{c}_3(M_1, M_2)^T_{\Omega_2} \cdot \tau_{\Omega_2} \\ 0 \end{pmatrix}$$

One of the advantages of this approach is the fact that due to the separation of the meshes by the interface function we do not have to provide matching meshes at the interface. This is of particular interest when simulating electrical machines, since it allows for an arbitrary rotational displacement between the motor parts without remeshing.

V. Model Problem

The motor we want to simulate is depicted in Fig. 1. Material in grey is iron, yellow materials are magnets. Their magnetisation is parallel in the direction as indicated in the picture. There are 12 slots with 4 pole-pairs in the stator – the double layer winding is defined according to the scheme in the figure.

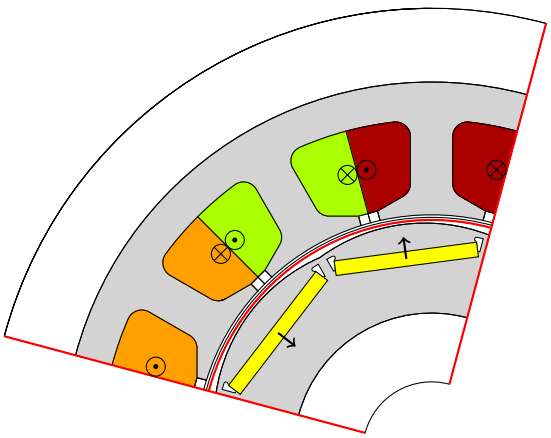

Figure 1. Motor sector (mortar edges depicted in red)

In case of electrical machines we use 3 different meshes; one for the rotor, one for the stator and one for the air gap domain. The rotor and the stator meshes are generated by NETGEN [9] resulting in an unstructured mesh, in the air gap we provide a structured mesh.

VI. Results

To compare the results computed by the finite element program at issue, the motor with the same specification was simulated with FEMAG [10].
FEMAG used a motor mesh with a total of 10552 nodes whereas the mesh for our program consited of 7799 nodes including 181 on the interface and 118 on the symmetry edges. The parameters in the interface integrals were chosen as $\alpha = 100$ and $\beta = 1$.
On the interface between rotor and air gap we get three different solution for the vector potential coming from the mesh on the rotor, one from the air gap mesh and additionaly the function λ on the interface. The differences of these solutions are shown in the following Fig. 2:

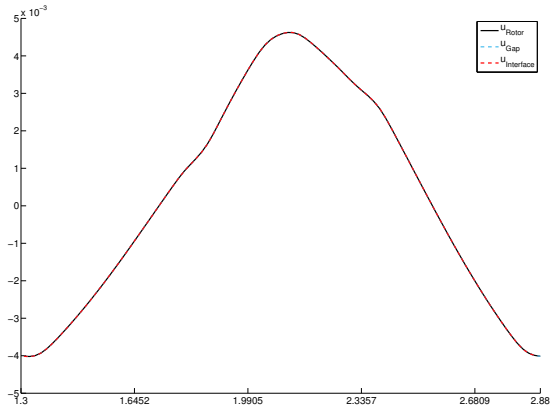

Figure 2. Comparison of the vectorpotential on the interface

In a postprocessing step the torque and linked fluxes are calculated. A comparison of these quantities with those estimated by FEMAG is depicted in Fig. 3 and 4 respectively.

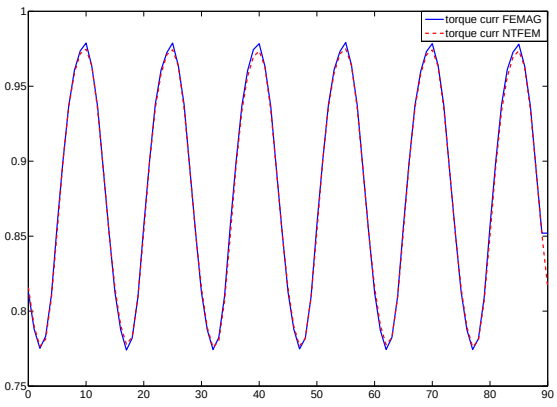

Figure 3. Torque comparision for the motor with currents

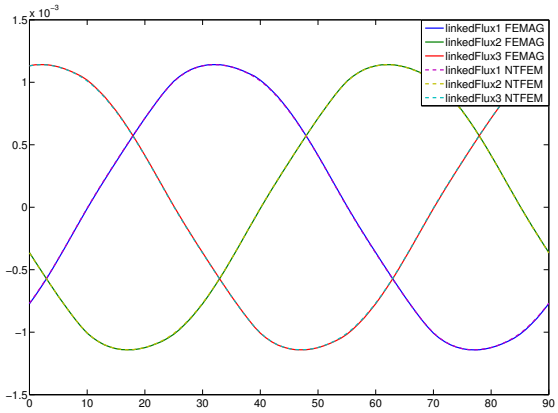

Figure 4. Linked flux comparision for the motor with currents

The differences in the curves of the torque vanish if we refine the mesh sufficiently.

VII. Future Work

Using any sort of domain decomposition technique enables us to use not only nonconforming meshes between the domains but as a result allows an arbitrary rotational displacement between rotor and stator. We will use this advantage in future work to construct more efficient path-following algorithms with respect to ϕ.

Acknowledgement

This work has been supported by the Austrian COMET-K2 program of the Linz Center of Mechatronics (LCM), and was funded by the Austrian federal government and the federal state of Upper Austria.

References

[1] B. Davat, Z. Ren, and M. Lajoie-Mazenc, "The movement in field modeling," *Magnetics, IEEE Transactions on*, vol. 21, no. 6, pp. 2296–2298, Nov 1985.

[2] E. Lange, F. Henrotte, and K. Hameyer, "A variational formulation for nonconforming sliding interfaces in finite element analysis of electric machines," *Magnetics, IEEE Transactions on*, vol. 46, no. 8, pp. 2755–2758, Aug 2010.

[3] B. Wohlmuth, "A mortar finite element method using dual spaces for the lagrange multiplier," *SIAM Journal on Numerical Analysis*, vol. 38, no. 3, pp. 989–1012, 2000. [Online]. Available: http://dx.doi.org/10.1137/S0036142999350929

[4] B. Rivière, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations*. Society for Industrial and Applied Mathematics, 2008. [Online]. Available: http://epubs.siam.org/doi/abs/10.1137/1.9780898717440

[5] J. Nitsche, "Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 36, no. 1, pp. 9–15, 1971.

[6] R. Stenberg, "Mortaring by a method of J.A. Nitsche," *Computational mechanics*, 1998.

[7] K. Pönitz, "Finite-Elemente-Mortaring nach einer Methode von J.A. Nitsche für elliptische Randwertaufgaben," Ph.D. dissertation, Chemnitz University of Technology.

[8] K. Hollaus, D. Feldengut, J. Schöberl, M. Wabro, and D. Omeragic, "Nitsche-type Mortaring for Maxwell's Equations," *Progress In Electromagnetics Research Symposium Proceedings*.

[9] Netgen. [Online]. Available: http://www.hpfem.jku.at/netgen/

[10] Femag. [Online]. Available: http://people.ee.ethz.ch/ femag/