Modeling and Motion Planning of the Magnetic Vector Potential in a Nonlaminated Active Thrust Bearing System

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Abstract

We propose a physical model representing a non laminated active thrust bearing system, obtained via Maxwell's equations, giving the magnetic vector potential with eddy currents. We show that this model is flat in a "practical" sense. In particular, motion planning of the magnetic flux and eddy currents can be easily deduced.

Keywords. Maxwell equations, eddy currents, flatness motion planning, active thrust bearings, vector potential

1 Introduction

As related by numerous authors [10], eddy currents in magnetic thrust bearings (MTB) affect sharply the magnitude and the dynamical response of magnetic forces.

The aim of this paper is to present a control solution able to settle a given magnetic flux in the air-gap, by imposing the coil voltage, in a prescribed duration while being able to influence the eddy current creation.

This problem constitutes a challenge not only since, according to Maxwell's partial differential equations [2], the system describing the dynamics of the magnetic flux in the MTB is inherently of an infinite-dimensional nature, but also because of the complexity of the geometry of the domain constituted by the bearing and the associated boundary conditions.

Our purpose is to show that the infinite dimensional system describing the vector potential in the MTB is flat in a "practical" sense, suitable to easily solve the above mentioned motion planning problem (see also e.g. [8] or [3, 4] in the context of the heat equation). For finite dimensional theoretic aspects and various applications of differential flatness the reader may refer to [5] (see also [6, 8] in the particular context of magnetic bearing systems).

The paper is organized as follows: Section 2 is devoted to the presentation of the infinite dimensional thrust bearing system controlled by its coil voltage. Its general solution is obtained,

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in Subsection 2.3.2, as a mixed series in trigonometric and Bessel functions (see e.g. [9] for an extensive introduction to Bessel functions). The notion of "pratical" flatness is then presented in Section 3, both in the time and frequency domains. The "pratical" flatness of our MTB system is studied in Subsection 3.2 and, finally, an application to motion planning is described in Section 4.

2 Description of the thrust bearing system

We derive a model for the vector potential in a solid MTB without slots. This model takes the form of a diffusion equation in several adjacent hollow cylinders (see Figure 1), with mixed Dirichlet-Robin-Neumann boundary conditions and controlled by its coil voltage, whose expression is given by:

$$U(t) = RI(t) + N\frac{d\Phi}{dt}(t) \tag{1}$$

with N the number of windings, R the resistance of the coil and $\Phi(t)$ the magnetic flux across the air gap.

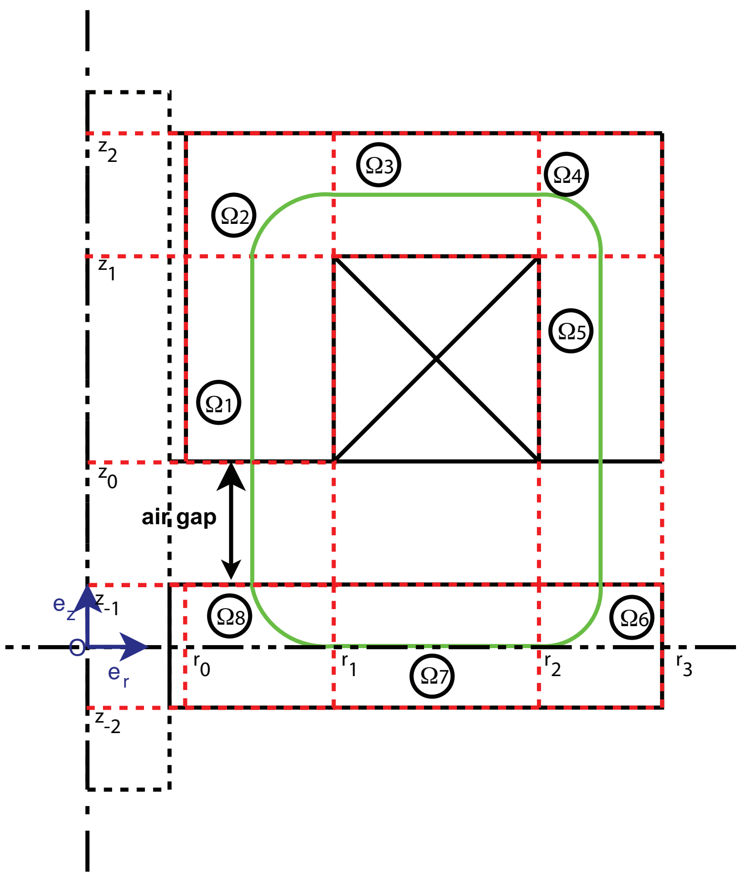

Figure 1: Sectional view of a half MTB

2.1 Choice of coordinates

Taking into account the symmetries of the problem, and in particular the invariance by rotation around the rotor axis, we use cylindrical coordinates centered at the rotor axis. The height coordinate is denoted by z and the radial one by r. Due to the curl invariance, the angular coordinate θ does not appear explicitly in the equations and is therefore omitted. We also denote the time by t.

According to the magnetic properties of this system, the vector potential is oriented along the ortho-radial direction. Hence it has only one non zero component in this direction, which simplifies the formulation of Maxwell's equations. Recall that the magnetic field and the induction are deduced from the vector potential by its curl, that the magnetic flux is obtained by a contour integral of the vector potential, and that the eddy currents are proportional to the time-derivative of the vector potential.

2.2 Maxwell's equations

In cylindrical coordinates (r,θ,z) with unit vectors $(\vec{e}_r,\vec{e}_\theta,\vec{e}_z)$, we consider a vector function $\vec{A}(r,\theta,z) = A_r(r,\theta,z)\vec{e}_r + A_\theta(r,\theta,z)\vec{e}_\theta + A_z(r,\theta,z)\vec{e}_z$. Recall that the vector Laplacian operator in cylindrical coordinates applied to a vector function $\vec{A}$ is given by:

$$\begin{aligned}\vec{\Delta}\vec{A} = &\left(\frac{\partial^2 A_r}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 A_r}{\partial \theta^2} + \frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r}\frac{\partial A_r}{\partial r} - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta}\right)\vec{e}_r \\ &+ \left(\frac{\partial^2 A_\theta}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{\partial^2 A_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r^2} - \frac{2}{r^2}\frac{\partial A_r}{\partial \theta}\right)\vec{e}_\theta \\ &+ \left(\frac{\partial^2 A_z}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \theta^2} + \frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r}\frac{\partial A_z}{\partial r} - \frac{A_z}{r^2}\right)\vec{e}_z\end{aligned} \tag{2}$$

If $\vec{A}$ is now a vector potential oriented along the ortho-radial direction and independent of θ, as it is the case in this paper, the only non zero component of $\vec{A}$ is A_θ which we denote by A from now on, for simplicity's sake. The previous expression (2) thus reads:

$$\vec{\Delta}\vec{A} = \left(\frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} - \frac{A}{r^2}\right)\vec{e}_\theta \tag{3}$$

We also denote by ΔA the component of $\vec{\Delta}\vec{A}$ with respect to e_θ, i.e.

$$\Delta A = \frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} - \frac{A}{r^2} \tag{4}$$

Maxwell's equations of the MTB system may be decomposed into 8 open subdomains, $\Omega_1,\dots,\Omega_8$, shown in Figure 1, with conductivity σ_i and magnetic permeability μ_i. In every open subdomain Ω_i, $i = 1,\dots,8$, we consider the corresponding vector potential, denoted by A_i. It satisfies

$$\Delta A_i(r,z,t) = \sigma_i\mu_i\frac{\partial A_i}{\partial t}(r,z,t), \forall t \in [0,T]\ , \forall (r,z) \in \Omega_i, \quad A(r,z,t) \triangleq \sum_{i=1}^{8} A_i(r,z,t)\mathbf{1}_{\Omega_i}(r,z) \tag{5}$$

where $\mathbf{1}_{\Omega_i}(\cdot,\cdot)$ is the indicator function of the subdomain Ω_i

We now detail the boundary conditions for all these subdomains. They are of three kinds: zero magnetic flux or zero vector potential on an edge, continuity of the vector potential and its gradient through an edge, or imposed tangential magnetic induction on the edges of the coil.

In the whole domain Ω, the initial condition is zero. The boundary conditions below are stated for all t:

- For $\partial\Omega_1$:

$$\begin{cases} A_1(r_0,z,t)=0 & \forall\, z\in]z_0,z_1[\\ \frac{1}{r_1}\frac{\partial}{\partial r}\left(rA_1(r,z,t)\right)\Big|_{r=r_1}=B_{T,1}(z,t) & \forall\, z\in]z_0,z_1[\\ \frac{\partial}{\partial z}A_1(r,z,t)\Big|_{z=z_0}=0 & \forall r\in]r_0,r_1[\\ A_1(r,z_1,t)=A_2(r,z_1,t) & \forall r\in]r_0,r_1[\end{cases} \tag{6}$$

- For $\partial\Omega_2$:

$$\begin{cases} A_2(r_0,z,t)=0 & \forall\, z\in]z_1,z_2[\\ \frac{1}{r_1}\frac{\partial}{\partial r}\left(rA_2(r,z,t)\right)\Big|_{r=r_1}=\frac{1}{r_1}\frac{\partial}{\partial r}\left(rA_3(r,z,t)\right)\Big|_{r=r_1} & \forall\, z\in]z_1,z_2[\\ \frac{\partial}{\partial z}A_2(r,z,t)\Big|_{z=z_1}=\frac{\partial}{\partial z}A_1(r,z,t)\Big|_{z=z_1} & \forall r\in]r_0,r_1[\\ A_2(r,z_2,t)=0 & \forall r\in]r_0,r_1[\end{cases} \tag{7}$$

- For $\partial\Omega_3$:

$$\begin{cases} A_3(r_1,z,t)=A_2(r_1,z,t) & \forall\, z\in]z_1,z_2[\\ A_3(r_2,z,t)=A_4(r_2,z,t) & \forall\, z\in]z_1,z_2[\\ \frac{\partial}{\partial z}\left(A_3(r,z,t)\right)\Big|_{z=z_1}=B_{T,3}(r,t) & \forall r\in]r_1,r_2[\\ A_3(r,z_2,t)=0 & \forall r\in]r_1,r_2[\end{cases} \tag{8}$$

- For $\partial\Omega_4$:

$$\begin{cases} A_4(r_3,z,t)=0 & \forall\, z\in]z_1,z_2[\\ \frac{1}{r_2}\frac{\partial}{\partial r}\left(rA_4(r,z,t)\right)\Big|_{r=r_2}=\frac{1}{r_2}\frac{\partial}{\partial r}\left(rA_3(r,z,t)\right)\Big|_{r=r_2} & \forall\, z\in]z_1,z_2[\\ \frac{\partial}{\partial z}A_4(r,z,t)\Big|_{z=z_1}=\frac{\partial}{\partial z}A_5(r,z,t)\Big|_{z=z_1} & \forall r\in]r_2,r_3[\\ A_4(r,z_2,t)=0 & \forall r\in]r_2,r_3[\end{cases} \tag{9}$$

- For $\partial\Omega_5$:

$$\begin{cases} A_5(r_3,z,t)=0 & \forall\, z\in]z_0,z_1[\\ \frac{1}{r_2}\frac{\partial}{\partial r}\left(rA_5(r,z,t)\right)\Big|_{r=r_2}=B_{T,5}(z,t) & \forall\, z\in]z_0,z_1[\\ \frac{\partial}{\partial z}A_5(r,z,t)\Big|_{z=z_0}=0 & \forall r\in]r_2,r_3[\\ A_5(r,z_1,t)=A_4(r,z_1,t) & \forall r\in]r_2,r_3[\end{cases} \tag{10}$$

- For $\partial\Omega_6$:

$$\begin{cases} A_6(r_0,z,t)=0 & \forall\, z\in]z_{-2},z_{-1}[\\ \frac{1}{r_1}\frac{\partial}{\partial r}\left(rA_6(r,z,t)\right)\Big|_{r=r_1}=\frac{1}{r_1}\frac{\partial}{\partial r}\left(rA_7(r,z,t)\right)\Big|_{r=r_1} & \forall\, z\in]z_{-2},z_{-1}[\\ A_6(r,z_{-1},t)=A_1(r,z_0,t) & \forall r\in]r_2,r_3[\\ A_6(r,z_{-2},t)=0 & \forall r\in]r_2,r_3[\end{cases} \tag{11}$$

- For $\partial\Omega_7$:

$$\begin{cases} A_7(r_1,z,t)=A_6(r_1,z,t) & \forall\, z\in]z_{-2},z_{-1}[\\ A_7(r_2,z,t)=A_8(r_2,z,t) & \forall\, z\in]z_{-2},z_{-1}[\\ \frac{\partial}{\partial z}\left(A_7(r,z,t)\right)\Big|_{z=z_{-1}}=B_{T,7}(r,t) & \forall r\in]r_1,r_2[\\ A_7(r,z_{-2},t)=0 & \forall r\in]r_1,r_2[\end{cases} \tag{12}$$

- For $\partial\Omega_8$:

$$\left\{\begin{array}{ll} A_8(r_3,z,t)=0 & \forall\, z\in]z_{-2},z_{-1}[\\ \frac{1}{r_2}\frac{\partial}{\partial r}\left(rA_8(r,z,t)\right)\Big|_{r=r_2} = \frac{1}{r_2}\frac{\partial}{\partial r}\left(rA_7(r,z,t)\right)\Big|_{r=r_2} & \forall\, z\in]z_{-2},z_{-1}[\\ A_8(r,z_{-1},t)=A_5(r,z_0,t) & \forall r\in]r_0,r_1[\\ A_8(r,z_{-2},t)=0 & \forall r\in]r_0,r_1[\end{array}\right. \tag{13}$$

where $B_{T,1}(z,t)$, $B_{T,3}(z,t)$, $B_{T,5}(z,t)$ and $B_{T,7}(z,t)$ are the tangential component of the magnetic induction created on the edges of the coil. It may be shown using [1], that there exist functions $Z_i,\ i=1,5$ and $R_j,\ j=3,7$, depending on the air gap l_e, such that:

$$\begin{array}{l} B_{T,i}(z,t)=Z_i(z,l_e)I(t) \\ B_{T,j}(r,t)=R_j(r,l_e)I(t) \end{array}$$

where $I(t)$ is the current intensity in the coil.

2.3 Solution in the particular case of three adjacent domains

2.3.1 The system of PDEs for three adjacent domains

For the sake of simplicity, we only present the solution in a set of three adjacent domains (say domains Ω_1, Ω_2 and Ω_3). The computation of the general solution follows the same lines.

We introduce the time Laplace transform of $A_i(r,z,t)$, $i\in 1,2,3$, noted $\hat{A}_i(r,z,s)\triangleq\int_0^{+\infty}A_i(r,z,t)e^{st}dt$. It is well known that the Laplace transform of $\frac{\partial A_i}{\partial t}$ is equal to $s\hat{A}_i(r,z,s)$.

For convenience, using the linearity of the diffusion equation, the vector potential is searched in the form:

$$A(r,z,t)\triangleq\sum_{i=1}^{3}A_i(r,z,t)\mathbf{1}_{\Omega_i}(r,z)\triangleq\sum_{i=1}^{3}\left(A_{ir}(r,z,t)+A_{iz}(r,z,t)\right)\mathbf{1}_{\Omega_i}(r,z) \tag{14}$$

where:

$$\hat{A}_{i,r}(r,z,s)=\sum_{k=0}^{+\infty}\alpha_{i,k}(s)\varphi_{r,i,k}(r,s)\psi_{r,i,k}(z) \tag{15}$$

$$\hat{A}_{i,z}(r,z,s)=\sum_{k=0}^{+\infty}\beta_{i,k}(s)\varphi_{z,i,k}(r)\psi_{z,i,k}(z,s) \tag{16}$$

$\hat{A}_{i,r}(r,z,s)$ and $\hat{A}_{i,z}(r,z,s)$ being solution respectively of

$$\Delta\hat{A}_{i,\kappa}(r,z,s)=\sigma_b\mu_b s\hat{A}_{i,\kappa}(r,z,s) \tag{17}$$

in Ω_i, $i=1,2,3$, $\kappa=r,z$, with boundary conditions:

- for $\partial\Omega_1$

$$\left\{\begin{array}{ll} \hat{A}_{1,r}(r_0,z,s)=0 & \forall\, z\in]z_0,z_1[\\ \frac{1}{r_1}\frac{\partial}{\partial r}\left(r\hat{A}_{1,r}(r,z,s)\right)\Big|_{r=r_1}=\hat{B}_{T,1}(z,s) & \forall\, z\in]z_0,z_1[\\ \frac{\partial}{\partial z}\hat{A}_{1,r}(r,z,s)\Big|_{z=z_0}=0 & \forall r\in]r_0,r_1[\\ \hat{A}_{1,r}(r,z_1,s)=0 & \forall r\in]r_0,r_1[\end{array}\right.$$

$$\left\{\begin{array}{ll} \hat{A}_{1,z}(r_0,z,s)=0 & \forall\, z\in]z_0,z_1[\\ \frac{1}{r_1}\frac{\partial}{\partial r}\left(r\hat{A}_{1,z}(r,z,s)\right)\Big|_{r=r_1}=0 & \forall\, z\in]z_0,z_1[\\ \frac{\partial}{\partial z}\hat{A}_{1,z}(r,z,s)\Big|_{z=z_0}=0 & \forall r\in]r_0,r_1[\\ \hat{A}_{1,z}(r,z_1,s)=\hat{A}_2(r,z_1,s) & \forall r\in]r_0,r_1[\end{array}\right. \tag{18}$$

- for $\partial\Omega_2$

$$\left\{\begin{array}{ll} \hat{A}_{2,r}(r_0,z,s)=0 & \forall\, z\in]z_1,z_2[\\ \frac{1}{r_1}\frac{\partial}{\partial r}\left(r\hat{A}_{2,r}(r,z,s)\right)\Big|_{r=r_1}=\frac{1}{r_1}\frac{\partial}{\partial r}\left(r\hat{A}_{3}(r,z,s)\right)\Big|_{r=r_1} & \forall\, z\in]z_1,z_2[\\ \frac{\partial}{\partial z}\hat{A}_{2,r}(r,z,s)\Big|_{z=z_1}=0 & \forall r\in]r_0,r_1[\\ \hat{A}_{2,r}(r,z_2,s)=0 & \forall r\in]r_0,r_1[\end{array}\right.$$

$$\left\{\begin{array}{ll} \hat{A}_{2,z}(r_0,z,s)=0 & \forall\, z\in]z_1,z_2[\\ \frac{1}{r_1}\frac{\partial}{\partial r}\left(r\hat{A}_{2,z}(r,z,s)\right)\Big|_{r=r_1}=0 & \forall\, z\in]z_1,z_2[\\ \frac{\partial}{\partial z}\hat{A}_{2,z}(r,z,s)\Big|_{z=z_1}=\frac{\partial}{\partial z}\hat{A}_{1}(r,z,s)\Big|_{z=z_1} & \forall r\in]r_0,r_1[\\ \hat{A}_{2,z}(r,z_2,s)=0 & \forall r\in]r_0,r_1[\end{array}\right. \tag{19}$$

- for $\partial\Omega_3$

$$\left\{\begin{array}{ll} \hat{A}_{3,r}(r_1,z,s)=\hat{A}_2(r_1,z,s) & \forall\, z\in]z_1,z_2[\\ \hat{A}_{3,r}(r_2,z,s)=\hat{A}_4(r_2,z,s) & \forall\, z\in]z_1,z_2[\\ \frac{\partial}{\partial z}\left(\hat{A}_{3,r}(r,z,s)\right)\Big|_{z=z_1}=0 & \forall r\in]r_1,r_2[\\ \hat{A}_{3,r}(r,z_2,s)=0 & \forall r\in]r_1,r_2[\end{array}\right.$$

$$\left\{\begin{array}{ll} \hat{A}_{3,z}(r_1,z,s)=0 & \forall\, z\in]z_1,z_2[\\ \hat{A}_{3,z}(r_2,z,s)=0 & \forall\, z\in]z_1,z_2[\\ \frac{\partial}{\partial z}\left(\hat{A}_{3,z}(r,z,s)\right)\Big|_{z=z_1}=\hat{B}_{T,3}(z,s) & \forall r\in]r_1,r_2[\\ \hat{A}_{3,z}(r,z_2,s)=0 & \forall r\in]r_1,r_2[\end{array}\right. \tag{20}$$

where $\hat{A}_4(r_2,z,s)$ is a given function.

Note that the components $\hat{A}_{i,r}(r,z,s)$ (resp. $\hat{A}_{i,z}(r,z,s)$) represent the radial (resp. axial) diffusion in the material.

2.3.2 Solution

Combining (15), (16) and (17), it is readily seen that $\varphi_{i,\kappa,k}$ and $\psi_{i,\kappa,k}$ are given by:

$$\frac{\frac{\partial^2\varphi_{i,r,k}}{\partial r^2}(r,s)+\frac{1}{r}\frac{\partial\varphi_{i,r,k}}{\partial r}(r,s)-\left(\sigma_b\mu_b s+\frac{1}{r^2}\right)\varphi_{i,r,k}(r,s)}{\varphi_{i,r,k}(r,s)}=-\frac{\frac{\partial^2\psi_{i,r,k}}{\partial z^2}(z)}{\psi_{i,r,k}(z)}=\lambda_{r,i,k}\ \forall k\in\mathbb{N}, i=1,2,3$$

$$-\frac{\frac{\partial^2\varphi_{i,z,k}}{\partial r^2}(r)+\frac{1}{r}\frac{\partial\varphi_{i,z,k}}{\partial r}(r)-\frac{1}{r^2}\varphi_{i,z,k}(r)}{\varphi_{i,z,k}(r)}=\frac{\frac{\partial^2\psi_{i,z,k}}{\partial z^2}(z,s)-\sigma_b\mu_b s\psi_{i,z,k}(z,s)}{\psi_{i,z,k}(z,s)}=\lambda_{z,i,k}\ \forall k\in\mathbb{N}, i=1,2,3$$

This leads to the two systems of equations:

$$\left\{\begin{array}{r}\frac{\partial^2\varphi_{i,r,k}}{\partial r^2}(r,s)+\frac{1}{r}\frac{\partial\varphi_{i,r,k}}{\partial r}(r,s)-\left(\sigma_b\mu_b s+\lambda_{r,i,k}+\frac{1}{r^2}\right)\varphi_{i,r,k}(r,s)=0\\ \frac{\partial^2\psi_{i,r,k}}{\partial z^2}(z)+\lambda_{r,i,k}\psi_{i,r,k}(z)=0\end{array}\right. \tag{21}$$

and

$$\left\{\begin{array}{r}\frac{\partial^2\varphi_{i,z,k}}{\partial r^2}(r)+\frac{1}{r}\frac{\partial\varphi_{i,z,k}}{\partial r}(r)+\left(\lambda_{z,i,k}-\frac{1}{r^2}\right)\varphi_{i,z,k}(r)=0\\ \frac{\partial^2\psi_{i,z,k}}{\partial z^2}(z,s)-(\lambda_{z,i,k}+\sigma_b\mu_b s)\psi_{i,z,k}(z,s)=0\end{array}\right. \tag{22}$$

Let us introduce the notations:

$$\left\{\begin{array}{l}\Xi_i(r_m,r_n,\eta)=\frac{I_i(r_n\eta)K_1(r_m\eta)+(-1)^iK_i(r_n\eta)I_1(r_m\eta))}{K_1(r_m\eta)}\\ \Upsilon_i(r_m,r_n,\lambda)=\frac{J_i\left(r_n\sqrt{\lambda}\right)Y_1\left(r_m\sqrt{\lambda}\right)-Y_i\left(r_n\sqrt{\lambda}\right)J_1\left(r_m\sqrt{\lambda}\right))}{Y_1\left(r_m\sqrt{\lambda}\right)}\end{array}\right. \tag{23}$$

with $r_m,r_n\in\mathbb{R}$, $\eta,\lambda\in\mathbb{C}$ and J_i, (resp. I_i) the Bessel function (resp. modified Bessel function) of the first kind and of the i-th order, $i=0,1$, and with Y_i, (resp. K_i) the Bessel function (resp. modified Bessel function) of the second kind and of the i-th order, $i=0,1$, [9].

In order to find the unknown functions corresponding to the direct coil contribution, we project $\hat{B}_{T,i}(z,s),\ i=1,3$ on the functional basis $\{\cos\left((z-z_0)\sqrt{\lambda_{r,1,k}}\right)|k\in\mathbb{N}\}$ and $\hat{B}_{T,j}(r,s),\ j=3,7$ on the functional basis $\{\Upsilon_1(r_1,r,\lambda_{z,3,k})|k\in\mathbb{N}\}$:

$$\begin{array}{l}\hat{B}_{T,i}(z,s)=Z_i(z,l_e)\hat{I}(s)=\hat{I}(s)\sum_{k=0}^{+\infty}\tilde{Z}_{i,k}(\hat{l}_e(s))\cos\left((z-z_0)\sqrt{\lambda_{r,i,k}}\right)\\ \hat{B}_{T,j}(r,s)=R_j(r,l_e)\hat{I}(s)=\hat{I}(s)\sum_{k=0}^{+\infty}\tilde{R}_{j,k}(\hat{l}_e(s))\Upsilon_1(r_1,r,\lambda_{z,j,k})\end{array} \tag{24}$$

where, $\forall k\in\mathbb{N}$, for $i=1,5$ and $j=3,7$:

$$\begin{array}{l}\tilde{Z}_{i,k}(l_e)=\frac{2}{z_1-z_0}\int_{z_0}^{z_1}Z_i(\xi,l_e)\cos\left((\xi-z_0)\sqrt{\lambda_{r,i,k}}\right)d\xi\\ \tilde{R}_{j,k}(l_e)=\frac{\int_{r_1}^{r_2}\rho R_j(\rho,l_e)\Upsilon_1(r_1,\rho,\lambda_{z,j,k})d\rho}{\int_{r_1}^{r_2}\rho\left(\Upsilon_1(r_1,\rho,\lambda_{z,j,k})\right)^2d\rho}\end{array} \tag{25}$$

Then, the solutions in Ω_i, $i=1,2,3$ are given, $\forall n\in\mathbb{N}$, by:

$$\left\{\begin{array}{l}\hat{A}_{1r}(r,z,s)=\hat{I}(s)\sum_{k=0}^{+\infty}\tilde{Z}_{1,k}(\hat{l}_e(s))\frac{\Xi_1(r_0,r,\Lambda_{r,1,k}(s))}{\Lambda_{r,1,k}(s)\Xi_0(r_0,r_1,\Lambda_{r,1,k}(s))}\cos\left((z-z_0)\sqrt{\lambda_{r,1,k}}\right)\\ \hat{A}_{1z}(r,z,s)=\sum_{k=0}^{+\infty}b_{1,k}(s)\Upsilon_1(r_0,r,\lambda_{z,1,k})\cosh\left((z-z_0)\Lambda_{z,1,k}(s)\right)\\ \hat{A}_{2r}(r,z,s)=\sum_{k=0}^{+\infty}a_{2,k}(s)\Xi_1(r_0,r,\Lambda_{r,2,k}(s))\sin\left((z_2-z)\sqrt{\lambda_{r,2,k}}\right)\\ \hat{A}_{2z}(r,z,s)=\sum_{k=0}^{+\infty}b_{2,k}(s)\Upsilon_1(r_0,r,\lambda_{z,1,k})\sinh\left((z_2-z)\Lambda_{z,1,k}(s)\right)\\ \hat{A}_{3r}(r,z,s)=\sum_{k=0}^{+\infty}\left(\gamma_{3,k}(s)I_1\left(r\Lambda_{r,2,k}(s)\right)+\delta_{3,k}(s)K_1\left(r\Lambda_{r,2,k}(s)\right)\right)\sin\left((z_2-z)\sqrt{\lambda_{r,2,k}}\right)\\ \hat{A}_{3z}(r,z,s)=\hat{I}(s)\sum_{k=0}^{+\infty}\tilde{R}_{3,k}(\hat{l}_e(s))\Upsilon_1(r_1,r,\lambda_{z,3,k})\frac{\sinh\left((z_2-z)\Lambda_{r,2,k}(s)\right)}{\Lambda_{r,2,k}(s)\cosh\left((z_2-z_1)\Lambda_{r,2,k}(s)\right)}\end{array}\right. \tag{26}$$

where, for $\kappa = r,z$, $i = 1\dots 8$ and $\forall k \in N$, $\lambda_{\kappa,i,k}$ and $\Lambda_{\kappa,i,k}(s)$ are defined in Appendix A.1 and where $b_{1,k}$, $a_{2,k}(s)$, $b_{2,k}(s)$, $\gamma_{3,k}(s)$, $\delta_{3,k}(s)$, are to be determined by the following system, deduced from the continuity of the vector potential and its gradient on the common edges of the adjacent domains:

$$\left\{\begin{array}{l}
b_{1,n}(s)\cosh((z_1-z_0)\Lambda_{z,1,n}(s)) - b_{2,n}(s)\sinh((z_2-z_1)\Lambda_{z,1,n}(s)) - \sum_{k=0}^{+\infty} a_{2,k}(s)h_{1,2,k,n}(s) = 0\\
b_{2,n}(s)\cosh((z_2-z_1)\Lambda_{z,1,n}(s)) + b_{1,n}(s)\sinh((z_1-z_0)\Lambda_{z,1,n}(s)) = \hat{I}(s)g_{1,2,n}(s)\\
(-1)^n\Lambda_{r,2,n}(s)a_{2,n}(s) - \Lambda_{z,3,n}(s)\left(\gamma_{3,n}I_0(r_1\Lambda_{z,3,n}(s)) - \delta_{3,n}K_0(r_1\Lambda_{z,3,n}(s))\right) = \hat{I}(s)g_{3,2,n}(s)\\
\gamma_{3,n}(s)I_1(r_1\Lambda_{z,1,n}(s)) + \delta_{3,n}(s)K_1(r_1\Lambda_{z,1,n}(s)) - a_{2,n}(s)\Xi_1(r_0,r_1,\Lambda_{r,2,n}(s)) - \sum_{k=0}^{+\infty} b_{2,k}(s)h_{2,3,k,n}(s) = 0\\
\gamma_{3,n}(s)I_1(r_2\Lambda_{z,1,n}(s)) + \delta_{3,n}(s)K_1(r_2\Lambda_{z,1,n}(s)) = g_{3,4,n}(s)
\end{array}\right. \tag{27}$$

with:

$$g_{3,4,n}(s) = \frac{2}{z_2-z_1}\int_{z_1}^{z_2} A_4(r_2,\xi,s)\sin\left((z_2-\xi)\sqrt{\lambda_{r,2,k}}\right)d\xi$$

and with $g_{1,2,n}(s)$, $g_{3,2,n}(s)$, $h_{1,2,k,n}(s)$, $h_{2,3,k,n}(s)$ defined in Appendix A.1 for the 8 sub domains problem. Suitably grouping theses equations, the system reads:

$$\begin{pmatrix}\mathscr{B}_{1,n} & -\mathscr{B}_{2,n}\\ \mathscr{B}_{3,n} & \mathscr{B}_{4,n}\end{pmatrix}\begin{pmatrix}b_{1,n}(s)\\ b_{2,n}(s)\end{pmatrix} = \begin{pmatrix}\sum_{k=0}^{+\infty} a_{2,k}(s)h_{1,2,k,n}(s)\\ g_{1,2,n}(s)\end{pmatrix} \tag{28}$$

with

$$\begin{array}{ll}
\mathscr{B}_{1,n} = \cosh((z_1-z_0)\Lambda_{z,1,n}(s)), & \mathscr{B}_{2,n} = \sinh((z_2-z_1)\Lambda_{z,1,n}(s))\\
\mathscr{B}_{3,n} = \sinh((z_1-z_0)\Lambda_{z,1,n}(s)), & \mathscr{B}_{4,n} = \cosh((z_2-z_1)\Lambda_{z,1,n}(s))
\end{array}$$

and

$$\begin{pmatrix}\Gamma_{1,n} & -\Gamma_{2,n}\\ \Gamma_{3,n} & \Gamma_{4,n}\end{pmatrix}\begin{pmatrix}\gamma_{3,n}(s)\\ \delta_{3,n}(s)\end{pmatrix} = \begin{pmatrix}(-1)^{n+1}\Lambda_{r,2,n}(s)a_{2,n}(s) + g_{3,2,n}(s)\\ a_{2,n}(s)\Xi_1(r_0,r_1,\Lambda_{r,2,n}(s)) + \sum_{k=0}^{+\infty} b_{2,k}(s)h_{2,3,k,n}(s)\end{pmatrix} \tag{29}$$

with

$$\begin{array}{ll}
\Gamma_{1,n} = \Lambda_{z,3,n}(s)I_0(r_1\Lambda_{z,3,n}(s)), & \Gamma_{2,n} = \Lambda_{z,3,n}(s)K_0(r_1\Lambda_{z,3,n}(s))\\
\Gamma_{3,n} = I_1(r_1\Lambda_{z,1,n}(s)), & \Gamma_{4,n} = K_1(r_1\Lambda_{z,1,n}(s))
\end{array}$$

It can be shown that the matrices $\mathscr{B}$ of (28) and Γ of (29) are invertible. Therefore, we can obtain $b_{1,n}$, $b_{2,n}$, $\gamma_{3,n}$ and $\delta_{3,n}$ in function of the $a_{2,n}$'s. Substituting them into the last equation of (27) for all $n \in \mathbb{N}$, we get a linear equation for the $a_{2,j}$'s for every $n \in \mathbb{N}$. Therefore, the complete solution of (17) with boundary conditions (18)-(20) is given by (14). Moreover, one can show that the coefficients $a_{2,j}$, previously obtained are linear with respect to $\hat{I}$, assuming that $g_{3,4,n}$ is linear with respect to $\hat{I}$. Therefore, we can define $a'_{2,n}$, $b'_{1,n}$, $b'_{2,n}$, $\gamma'_{3,n}$ and $\delta'_{3,n}$ such that $\forall n \in \mathbb{N}$:

$$\begin{array}{l}
a_{2,n}(s) = \hat{I}(s)a'_{2,n}(s), \quad b_{1,n}(s) = \hat{I}(s)b'_{1,n}(s), \quad b_{2,n}(s) = \hat{I}(s)b'_{2,n}(s),\\
\gamma_{3,n}(s) = \hat{I}(s)\gamma'_{3,n}(s), \quad \text{and} \quad \delta_{3,n}(s) = \hat{I}(s)\delta'_{3,n}(s)
\end{array} \tag{30}$$

The solution of System (5)-(13) is obtained in an analogous way (see Appendix A.1). It may also be proven that the analogue of the coefficients $a_{2,j}$, $b_{1,n}$, $b_{2,n}$, $\gamma_{3,n}$ and $\delta_{3,n}$ are linear with respect to $\hat{I}$.

3 Differential flatness

3.1 Notion of “practical” flatness for infinite-dimensional systems

Basics on finite dimensional differential flatness may be found in [5] (see also [6] in the context of magnetic bearing systems). Roughly speaking, a system with m inputs u and n states x is differentially flat if there exists an m-dimensional output y (called flat output), with functionally independent components, function of x, u, and possibly a **finite** number of time derivatives of u, for which x and u can be expressed as functions of y and a **finite** number of its time derivatives.

In our problem, the model with state $\vec{A}$ and input U is **infinite**-dimensional, and the above definition does not directly apply. Here, we do not want to make a rigorous theory of flatness for infinite dimensional systems and we restrict to an *ad hoc* definition, well suited to obtain a solution of the motion planning problem. This is why this property is called *practical*.

3.1.1 In the time domain

Let us denote by $\Omega = \bigcup_{i=1}^{8} \Omega_i$. The set of infinitely differentiable functions from $[0,T]$ to $\mathbb{R}$ (resp. from $\Omega \times [0,T]$) to $\mathbb{R}$) is denoted by $C^\infty([0,T])$ (resp. $C^\infty(\Omega \times [0,T])$). We consider the system Σ defined by (5)-(13).

Definition 1. *We say that Σ is “practically” differentially flat (or, in short, flat) if there exist linear operators*

- $\mathscr{A} : C^\infty([0,T]) \longrightarrow C^\infty(\Omega \times [0,T])$
- $\mathscr{U} : C^\infty([0,T]) \longrightarrow C^\infty([0,T])$
- $\mathscr{Y} : C^\infty(\Omega \times [0,T]) \times C^\infty([0,T]) \longrightarrow C^\infty([0,T])$

such that if $A \in C^\infty(\Omega \times [0,T])$ is solution of Σ corresponding to $U \in C^\infty([0,T])$, then

$$y = \mathscr{Y}(A,U) = \mathscr{Y}_A A + \mathscr{Y}_U U \Longleftrightarrow \mathscr{A} y = A,\ \mathscr{U} y = U.$$

3.1.2 In the frequency domain

Assuming that the Laplace transforms of the signals A, U and y are well defined for all $s \in \sqrt{-1}\mathbb{R}$, in the frequency domain and denoting by $\hat{A}, \hat{U}, \hat{y}$ the Laplace transforms of A, U, y respectively, Definition 2 reads[1]:

Definition 2. *We say that Σ is “practically” differentially flat (or, in short, flat) if there exist linear operators*

- $\overline{\mathscr{A}} : C^\infty(\sqrt{-1}\,\mathbb{R}) \longrightarrow C^\infty(\Omega \times \sqrt{-1}\,\mathbb{R})$
- $\overline{\mathscr{U}} : C^\infty(\sqrt{-1}\,\mathbb{R}) \longrightarrow C^\infty(\sqrt{-1}\,\mathbb{R})$
- $\overline{\mathscr{Y}} : C^\infty(\Omega \times \sqrt{-1}\,\mathbb{R}) \times C^\infty(\sqrt{-1}\,\mathbb{R}) \longrightarrow C^\infty(\sqrt{-1}\,\mathbb{R})$

such that if $\hat{A} \in C^\infty(\Omega \times \sqrt{-1}\,\mathbb{R})$ is solution of Σ corresponding to $\hat{U} \in C^\infty(\sqrt{-1}\,\mathbb{R})$, then

$$\hat{y} = \overline{\mathscr{Y}}(\hat{A},\hat{U}) = \overline{\mathscr{Y}}_A \hat{A} + \overline{\mathscr{Y}}_U \hat{U} \Longleftrightarrow \overline{\mathscr{A}} \hat{y} = \hat{A},\ \overline{\mathscr{U}} \hat{y} = \hat{U}.$$

Remark 1. *In Definition 2, the operators $\overline{\mathscr{A}}$, $\overline{\mathscr{U}}$ and $\overline{\mathscr{Y}}$ are not necessarily equal to $\mathscr{A}$, $\mathscr{U}$ and $\mathscr{Y}$ respectively, of Definition 1, since the latter operators may depend on time.*

[1] Definition 2 can be made rigorous by using Mikusiński’s operational calculus [7]

3.2 A flat output

The line integral of the tangential component of the magnetic induction on one edge of the MTB is, according to Maxwell's equations, the line integral of the normal derivative of the vector potential on this edge. Choosing the edge of Ω_1 opposed to the coil, the time function denoted by y, given by the line integral of the tangential magnetic induction on this edge is our candidate flat output:

$$\hat{y}(s) = \overline{\mathscr{Y}}(\hat{A}) = \frac{1}{r_0}\int_{z_0}^{z_1} \frac{\partial}{\partial r}\left(r\hat{A}_1(r,\xi,s)\right)\Big|_{r=r_0} d\xi \tag{31}$$

where A_1 is defined in (14).

Let us show that the knowledge of this output allows to obtain the value of the vector potential in the whole domain Ω and the value of the corresponding coil voltage.

Formula (31), with (26) and (27), combined with Wronskian identities of Bessel functions [9], reads:

$$\hat{y}(s) = \frac{2\hat{I}(s)}{r_0}\left(\sum_{k=0}^{\infty}\frac{(-1)^k\tilde{Z}_{1,k}(\hat{l}_e(s))}{\Lambda_{r,1,k}(s)\left(\frac{\pi}{2}+k\pi\right)\Xi_0(r_0,r_1,\Lambda_{r,1,k}(s))} + \frac{1}{\pi}\sum_{k=0}^{+\infty}\frac{b'_{1,k}(s)\sinh((z_1-z_0)\Lambda_{z,1,k}(s))}{\Lambda_{z,1,k}(s)}\right) \tag{32}$$

with $b'_{1,k}(s) = \frac{b_{1,k}(s)}{\hat{I}(s)}$ (see (30)) and $b_{1,k}(s)$ given by (27).
We set

$$\hat{G}(s) \triangleq \frac{2}{r_0}\left(\sum_{k=0}^{\infty}\frac{(-1)^k\tilde{Z}_{1,k}(\hat{l}_e(s))}{\Lambda_{r,1,k}(s)\left(\frac{\pi}{2}+k\pi\right)\Xi_0(r_0,r_1,\Lambda_{r,1,k}(s))} + \frac{1}{\pi}\sum_{k=0}^{+\infty}\frac{b'_{1,k}(s)\sinh((z_1-z_0)\Lambda_{z,1,k}(s))}{\Lambda_{z,1,k}(s)}\right)$$

Remark that $\hat{G}(s)$ does not depend on $\hat{I}(s)$. Then:

$$\hat{I}(s) = \frac{\hat{y}(s)}{\hat{G}(s)} \tag{33}$$

wherever this fraction is well defined.

Thus, we can define $\overline{\mathscr{U}}$ (resp. $\overline{\mathscr{A}}$) using (1) (resp. (36) and (37)). $\overline{\mathscr{U}}$ reads:

$$\overline{\mathscr{U}}(s) = \frac{1}{\hat{G}(s)}\left(R + 2N\pi r_1 s\left(\frac{2}{z_1-z_0}\sum_{k=0}^{+\infty}\frac{\Xi_1(r_0,r_1,\Lambda_{r,1,k}(s))}{\Lambda_{r,1,k}(s)\Xi_0(r_0,r_1,\Lambda_{r,1,k}(s))}\tilde{Z}_{1,k} + \sum_{k=0}^{+\infty} b'_{1,k}(s)\Upsilon_1(r_0,r_1,\lambda_{z,1,k})\right)\right) \tag{34}$$

The expression of $\overline{\mathscr{A}}$ is obtained by combining (36) given in Section A.2, (37) and (33). Its expression, very long and complicated, is omitted due to the lack of space.
Therefore the vector potential in Ω and the corresponding coil voltage are everywhere given as functions of y according to the expression of $\overline{\mathscr{A}}$ and (34). It is also possible to deduce the value of the magnetic flux in function of y as shown in Appendix A.2.

4 Motion planning

In this section, the knowledge of the flat output is used to control, in open loop, the flux in the air gap between two given steady states: $\Phi^*(t_0)$ and $\Phi^*(t_1)$, in a prescribed duration $(t_1 - t_0)$, the rotor being kept fixed. The superscript "$*$" stands for reference trajectories.

When the flux is brought from one value to another in a finite time, eddy currents appear in the magnetic material of the MTB, thus resulting in an increase of the flux settling time and dissipated energy.

There is an infinite number of ways to bring the flux from one value to another, with more or less eddy currents. In order to reduce the eddy currents in the MTB, we can impose the variation of the flux using the flatness property of the system.

We follow the following steps:

1. We calculate the initial and final values of the flat output $y(t)$ from the values $\Phi^*(t_0)$ and $\Phi^*(t_1)$. For $i = 0, 1$:

$$y_i^* = y(t_i) = \frac{\hat{G}(0)\Phi^*(t_i)}{2\pi r_1 \left(\frac{2}{z_1 - z_0}\sum_{k=0}^{+\infty}\tilde{Z}_{1,k}(l_e(t))\frac{\Xi_1(r_0,r_1,\Lambda_{r,1,k}(0))}{\Lambda_{r,1,k}(0)\Xi_0(r_0,r_1,\Lambda_{r,1,k}(0))} + \sum_{k=0}^{+\infty} b'_{1,k}(0)\Upsilon_1(r_0,r_1,\lambda_{z,1,k})\right)}$$

2. We define the trajectory $y(t)$ between these two values as a Gevrey function (see [8] p.91):

$$y_{gev}(t) = \begin{cases} 0 & \text{for } t < 0 \\ \dfrac{\int_0^{\frac{t-t_0}{t_1-t_0}} e^{-(\tau(1-\tau))^{-\upsilon}} d\tau}{\int_0^1 e^{-(\tau(1-\tau))^{-\upsilon}} d\tau} & \text{for } 0 \le t \le t_1 \\ 1 & \text{for } t > t_1 \end{cases} \quad (35)$$

   where $\upsilon \geq 1$ is a real parameter tat may be tuned: the steepness of the function increases with υ. Thus y reads :

$$y(t) = (y_1^* - y_0^*)\, y_{gev}(t) + y_0^*$$

3. The corresponding voltage $U(t)$ is thus obtained using the inverse Laplace transform of $\overline{\mathscr{U}}(s)$ given in (34):

$$U(t) = \mathscr{L}^{-1}\left(\overline{\mathscr{U}}(s)\right) y(t)$$

   where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform, defined by

$$\mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi\iota}\lim_{T\to+\infty}\int_{c-\iota T}^{c+\iota T} e^{st}F(s)ds$$

   with $\iota = \sqrt{-1}$ and $c \in \mathbb{R}$, provided that the integral converges.

4. We compute the corresponding induced flux $\Phi_e(t)$, created by the eddy currents, using the inverse Laplace transform of $\overline{\mathscr{F}}(s)$ defined in Appendix A.2 and given by (41):

$$\Phi_e(t) = \mathscr{L}^{-1}\left(\overline{\mathscr{F}}(s)\right) y(t)$$

   Assuming that this flux is zero at $t = 0$, we may try to maintain its time derivative as small as possible in order to keep $\Phi_e(t)$ small, $\forall t \in [t_0, t_1]$.

Then, the computed voltage $U(t)$ to be applied to the coil is deduced.

5 Conclusion

A model of the MTB without slots and where eddy currents are taken into account has been derived in this paper. It has been shown that this infinite dimensional system is flat in a "practical" sense. The flat output chosen is the integral of the tangential component of the magnetic induction on the edge of Ω_1 at $r = r_0$. Finally, this flat output is used to drive the magnetic flux, in open loop, between two steady states in prescribed duration. The minimization of the influence of the eddy currents in the effective magnetic flux and the feedback regulation problem will be studied in future works.

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A Appendix

A.1 Expression of the vector potential in Ω as functions of $\hat{y}(s)$

The vector potential in the whole domain Ω_1 reads:

$$\left\{\begin{array}{l}
\hat{A}_{1r}(r,z,s) = \hat{I}(s)\sum_{k=0}^{+\infty}\tilde{Z}_{1,k}(\hat{l}_e(s))\frac{\Xi_1(r_0,r,\Lambda_{r,1,k}(s))}{\Lambda_{r,1,k}(s)\Xi_0(r_0,r_1,\Lambda_{r,1,k}(s))}\cos\left((z-z_0)\sqrt{\lambda_{r,1,k}}\right)\\
\hat{A}_{1z}(r,z,s) = \sum_{k=0}^{+\infty} b_{1,k}(s)\Upsilon_1(r_0,r,\lambda_{z,1,k})\cosh\left((z-z_0)\Lambda_{z,1,k}(s)\right)\\
\hat{A}_{2r}(r,z,s) = \sum_{k=0}^{+\infty} a_{2,k}(s)\Xi_1(r_0,r,\Lambda_{r,2,k}(s))\sin\left((z_2-z)\sqrt{\lambda_{r,2,k}}\right)\\
\hat{A}_{2z}(r,z,s) = \sum_{k=0}^{+\infty} b_{2,k}(s)\Upsilon_1(r_0,r,\lambda_{z,2,k})\sinh\left((z_2-z)\Lambda_{z,2,k}(s)\right)\\
\hat{A}_{3r}(r,z,s) = \sum_{k=0}^{+\infty}\left(\gamma_{3,k}(s)I_1\left(r\Lambda_{r,3,k}(s)\right)+\delta_{3,k}(s)K_1\left(r\Lambda_{r,3,k}(s)\right)\right)\sin\left((z_2-z)\sqrt{\lambda_{r,3,k}}\right)\\
\hat{A}_{3z}(r,z,s) = \hat{I}(s)\sum_{k=0}^{+\infty}\tilde{R}_{3,k}(\hat{l}_e(s))\Upsilon_1(r_1,r,\lambda_{z,3,k})\frac{\sinh\left((z_2-z)\Lambda_{r,3,k}(s)\right)}{\Lambda_{r,3,k}(s)\cosh\left((z_2-z_1)\Lambda_{r,3,k}(s)\right)}\\
\hat{A}_{4r}(r,z,s) = \sum_{k=0}^{+\infty} a_{4,k}(s)\Xi_1(r_3,r,\Lambda_{r,4,k}(s))\sin\left((z_2-z)\lambda_{r,4,k}\right)\\
\hat{A}_{4z}(r,z,s) = \sum_{k=0}^{+\infty} b_{4,k}(s)\Upsilon_1(r_3,r,\lambda_{z,4,k})\sinh\left((z_2-z)\Lambda_{z,4,k}(s)\right)\\
\hat{A}_{5r}(r,z,s) = \hat{I}(s)\sum_{k=0}^{+\infty}\tilde{Z}_{5,k}(\hat{l}_e(s))\frac{\Xi_1(r_3,r,\Lambda_{r,5,k}(s))}{\Lambda_{r,5,k}(s)\Xi_0(r_3,r_2,\Lambda_{r,5,k}(s))}\cos\left((z-z_0)\sqrt{\lambda_{r,5,k}}\right)\\
\hat{A}_{5z}(r,z,s) = \sum_{k=0}^{+\infty} b_{5,k}(s)\Upsilon_1(r_3,r,\lambda_{z,5,k})\cosh\left((z-z_0)\Lambda_{z,5,k}(s)\right)\\
\hat{A}_{6r}(r,z,s) = \sum_{k=0}^{+\infty} a_{6,k}(s)\Xi_1(r_3,r,\Lambda_{r,6,k}(s))\sin\left((z-z_{-2}(s))\sqrt{\lambda_{r,6,k}}\right)\\
\hat{A}_{6z}(r,z,s) = \sum_{k=0}^{+\infty} b_{6,k}(s)\Upsilon_1(r_3,r,\lambda_{z,6,k})\sinh\left((z-z_{-2}(s))\Lambda_{z,6,k}(s)\right)\\
\hat{A}_{7r}(r,z,s) = \sum_{k=0}^{+\infty}\left(\gamma_{7,k}(s)I_1\left(r\Lambda_{r,7,k}(s)\right)+\delta_{7,k}(s)K_1\left(r\Lambda_{r,7,k}(s)\right)\right)\sin\left((z-z_{-2}(s))\sqrt{\lambda_{r,7,k}}\right)\\
\hat{A}_{7z}(r,z,s) = \hat{I}(s)\sum_{k=0}^{+\infty}\tilde{R}_{7,k}(\hat{l}_e(s))\Upsilon_1(r_1,r,\lambda_{z,7,k})\frac{\sinh\left((z-z_{-2}(s))\Lambda_{r,7,k}(s)\right)}{\Lambda_{r,7,k}(s)\cosh\left(h_d\Lambda_{r,7,k}(s)\right)}\\
\hat{A}_{8r}(r,z,s) = \sum_{k=0}^{+\infty} a_{8,k}(s)\Xi_1(r_0,r,\Lambda_{r,8,k}(s))\sin\left((z-z_{-2}(s))\sqrt{\lambda_{r,8,k}}\right)\\
\hat{A}_{8z}(r,z,s) = \sum_{k=0}^{+\infty} b_{8,k}(s)\Upsilon_1(r_0,r,\lambda_{z,8,k})\sinh\left((z-z_{-2}(s))\Lambda_{z,8,k}(s)\right)
\end{array}\right. \tag{36}$$

with, $\forall k \in \mathbb{N}$:

- $\sqrt{\lambda_{r,1,k}} = \sqrt{\lambda_{r,5,k}} = \frac{\frac{\pi}{2}+k\pi}{z_1-z_0}, \quad \sqrt{\lambda_{r,2,k}} = \sqrt{\lambda_{r,3,k}} = \sqrt{\lambda_{r,4,k}} = \frac{\frac{\pi}{2}+k\pi}{z_2-z_1}, \quad k \in \mathbb{N}$
  $\sqrt{\lambda_{r,6,k}} = \sqrt{\lambda_{r,7,k}} = \sqrt{\lambda_{r,8,k}} = \frac{k\pi}{h_d}, \quad k \in \mathbb{N}^*$

- $\sqrt{\lambda_{z,1,k}}$, $\sqrt{\lambda_{z,2,k}}$ and $\sqrt{\lambda_{z,8,k}}$ are the positive roots of $\Upsilon_0(r_0,r_1,\lambda)=0, \quad k \in \mathbb{N}$

- $\sqrt{\lambda_{z,3,k}}$ and $\sqrt{\lambda_{z,7,k}}$ are the positive roots of $\Upsilon_1(r_1,r_2,\lambda)=0, \quad k \in \mathbb{N}$

- $\sqrt{\lambda_{z,4,k}}$, $\sqrt{\lambda_{z,5,k}}$ and $\sqrt{\lambda_{z,6,k}}$ are the positive roots of $\Upsilon_0(r_3,r_2,\lambda)=0, \quad k \in \mathbb{N}$

- $\forall k \in \mathbb{N}: \quad \begin{array}{l} \Lambda_{\kappa,i,k}(s) = \sqrt{\lambda_{\kappa,i,k} + \sigma_b \mu_b s},\ \kappa = r,z,\ i = 1,\cdots,5 \\ \Lambda_{\kappa,j,k}(s) = \sqrt{\lambda_{\kappa,j,k} + \sigma_d \mu_d s},\ \kappa = r,z,\ j = 6,7,8 \end{array}$

h_d is the width of the disc.The unknown functions $a_{m,k}(s),\ m = 2,4,6,8,\ b_{n,k}(s),\ n = 1,2,4,5,6,8,$ $\gamma_{p,k}(s)$ and $\delta_{p,k}(s),\ p = 3,7$, are to be determined by the following system, deduced from the continuity of the vector potential and its gradient on the common edges of the adjacent domains:

$$\left\{\begin{array}{l}
b_{1,n}(s)\cosh((z_1-z_0)\Lambda_{z,1,n}(s)) - b_{2,n}(s)\sinh((z_2-z_1)\Lambda_{z,2,n}(s)) - \sum_{k=0}^{+\infty} a_{2,k}(s)h_{1,2,k,n}(s) = 0 \\
\Lambda_{z,1,n}(s)\left(b_{2,n}(s)\cosh((z_2-z_1)\Lambda_{z,1,n}(s)) + b_{1,n}(s)sinh((z_1-z_0)\Lambda_{z,1,n}(s))\right) = \hat{I}(s)g_{12,n}(s) \\
a_{2,n}(s)\Lambda_{r,2,n}(s)\Xi_0(r_0,r_1,\Lambda_{r,2,n}(s)) - \Lambda_{r,3,n}(s)\left(\gamma_{3,n}I_0(r_1\Lambda_{r,3,n}(s)) - \delta_{3,n}K_0(r_1\Lambda_{r,3,n}(s))\right) = \hat{I}(s)g_{3,2,n}(s) \\
\gamma_{3,n}(s)I_1(r_1\Lambda_{r,3,n}(s)) + \delta_{3,n}(s)K_1(r_1\Lambda_{r,3,n}(s)) - a_{2,n}(s)\Xi_1(r_0,r_1,\Lambda_{r,2,n}(s)) - \sum_{k=0}^{+\infty} b_{2,k}(s)h_{2,3,k,n}(s) = 0 \\
\gamma_{3,n}(s)I_1(r_2\Lambda_{r,3,n}(s)) + \delta_{3,n}(s)K_1(r_2\Lambda_{r,3,n}(s)) - a_{4,n}(s)\Xi_1(r_3,r_2,\Lambda_{r,4,n}(s)) - \sum_{k=0}^{+\infty} b_{4,k}(s)h_{3,4,k,n}(s) = 0 \\
a_{4,n}\Lambda_{r,4,n}(s))\Xi_0(r_0,r_1,\Lambda_{r,4,n}(s)) - \Lambda_{r,3,n}(s)\left(\gamma_{3,n}I_0(r_2\Lambda_{r,3,n}(s)) - \delta_{3,n}K_0(r_2\Lambda_{r,3,n}(s))\right) = \hat{I}(s)g_{3,4,n}(s) \\
\Lambda_{z,4,n}(s)\left(b_{4,n}\cosh((z_2-z_1)\Lambda_{z,4,n}(s)) + b_{5,n}\sinh((z_1-z_0)\Lambda_{z,5,n}(s))\right) = \hat{I}(s)g_{5,4,n}(s) \\
b_{5,n}\cosh((z_1-z_0)\Lambda_{z,5,n}(s)) - b_{4,n}\sinh((z_2-z_1)\Lambda_{z,4,n}(s)) - \sum_{k=0}^{+\infty} a_{4,k}(s)h_{4,5,k,n}(s) = 0 \\
b_{6,n}(s)\sinh\left(h_d\Lambda_{z,6,n}(s)\right) - b_{5,n} = \hat{I}(s)g_{5,6,n}(s) \\
a_{6,n}(s)\Lambda_{r,6,n}(s)\Xi_0(r_3,r_2,\Lambda_{r,6,n}(s)) - \Lambda_{r,6,n}(s)\left(\gamma_{7,n}(s)I_0\left(r_2\Lambda_{r,6,n}(s)\right) - \delta_{7,n}(s)K_0\left(r_2\Lambda_{r,6,n}(s)\right)\right) = \hat{I}(s)g_{7,6,n}(s) \\
\left(\gamma_{7,n}I_1(r_2\Lambda_{r,7,n}(s)) + \delta_{7,n}K_1(r_2\Lambda_{r,7,n}(s))\right) - a_{6,n}\Xi_1(r_3,r_2,\Lambda_{r,6,n}(s)) - \sum_{k=0}^{+\infty} b_{6,k}(s)h_{6,7,k,n}(s) = 0 \\
\left(\gamma_{7,n}I_1(r_1\Lambda_{r,7,n}(s)) + \delta_{7,n}K_1(r_1\Lambda_{r,7,n}(s))\right) - a_{8,n}\Xi_1(r_0,r_1,\Lambda_{r,8,n}(s)) - \sum_{k=0}^{+\infty} b_{8,k}(s)h_{7,8,k,n}(s) = 0 \\
a_{8,n}(s)\Lambda_{r,8,n}(s)\Xi_0(r_0,r_1,\Lambda_{r,6,n}(s)) - \Lambda_{r,7,n}(s)\left(\gamma_{7,n}(s)I_0\left(r_1\Lambda_{r,7,n}(s)\right) - \delta_{7,n}(s)K_0\left(r_1\Lambda_{r,7,n}(s)\right)\right) = \hat{I}(s)g_{7,8,n}(s) \\
b_{8,n}(s)\cosh\left(h_d\Lambda_{z,8,n}(s)\right) - b_{1k}(s) = \hat{I}(s)g_{1,8,n}(s)
\end{array}\right. \tag{37}$$

Where:

$$
\begin{array}{l}
\nu_{1n} = \sum_{k=0}^{+\infty} \int_{r_0}^{r_1} \rho \left|\Upsilon_1(r_0,\rho,\lambda_{z,1,n}))\right|^2 d\rho \\
\nu_{2n} = \sum_{k=0}^{+\infty} \int_{r_2}^{r_3} \rho \left|\Upsilon_1(r_3,\rho,\lambda_{z,5,n}))\right|^2 d\rho \\
g_{1,2,n}(s) = \frac{1}{\nu_{1n}} \sum_{k=0}^{+\infty} \frac{(-1)^k \tilde{Z}_{1,k}(\hat{I}_e(s))\sqrt{\lambda_{r,1,k}}}{\Lambda_{r,1,k}(s)\Xi_0(r_0,r_1,\Lambda_{r,1,k}(s))} \int_{r_0}^{r_1} \rho \Xi_1(r_0,\rho,\Lambda_{r,1,k}(s))\Upsilon_1(r_0,\rho,\lambda_{z,1,n})d\rho \\
g_{3,2,n}(s) = \frac{4}{\pi r_1(z_2-z_1)} \sum_{k=0}^{+\infty} \tilde{R}_{3,k}(\hat{I}_e(s))\frac{1}{\Lambda_{r,2,k}(s)\cosh\left((z_2-z_1)\Lambda_{r,2,k}(s)\right)} \int_{z_1}^{z_2} \sinh\left((z_2-\xi)\Lambda_{r,2,k}(s)\right) \sin\left((z_2-\xi)\sqrt{\lambda_{r,2,n}}\right) d\xi \\
g_{3,4,n}(s) = \frac{2}{(z_2-z_1)} \sum_{k=0}^{+\infty} \tilde{R}_{3,k}(\hat{I}_e(s))\frac{\sqrt{\lambda_{z,3,k}}\Upsilon_0(r_1,r_2,\lambda_{z,3,k})}{\Lambda_{r,4,k}(s)\cosh\left((z_2-z_1)\Lambda_{r,4,k}(s)\right)} \int_{z_1}^{z_2} \sinh\left((z_2-\xi)\Lambda_{r,4,k}(s)\right) \sin\left((z_2-\xi)\sqrt{\lambda_{r,4,n}}\right) d\xi \\
g_{5,4,n}(s) = \frac{1}{\nu_{2n}} \sum_{k=0}^{+\infty} \frac{(-1)^k \tilde{Z}_{5,k}(\hat{I}_e(s))}{\Lambda_{r,5,k}(s)\Xi_0(r_3,r_2,\Lambda_{r,5,k}(s))} \int_{r_2}^{r_3} \rho \Xi_1(r_3,\rho,\Lambda_{r,5,k}(s))\Upsilon_1(r_3,\rho,\lambda_{z,5,n})d\rho \\
g_{7,6,n}(s) = \frac{2}{h_d} \sum_{k=0}^{+\infty} \tilde{R}_{7,k}(\hat{I}_e(s))\frac{\Upsilon_0(r_1,r_2,\lambda_{z,7,k})\lambda_{z,7,k}}{\Lambda_{r,6,k}(s)\cosh\left(h_d\Lambda_{r,6,k}(s)\right)} \int_{z_{-1}}^{z_{-2}} \sinh\left((\xi-z_{-2})\Lambda_{r,6,k}(s)\right) \sin\left((\xi-z_{-2})\sqrt{\lambda_{r,6,n}}\right) d\xi \\
g_{5,6,n}(s) = \frac{1}{\nu_{2n}} \sum_{k=0}^{+\infty} \frac{(-1)^k \tilde{Z}_{5,k}(\hat{I}_e(s))}{\Lambda_{r,5,k}(s)\Xi_0(r_3,r_2,\Lambda_{r,5,k}(s))} \int_{r_2}^{r_3} \rho \Xi_1(r_3,\rho,\Lambda_{r,5,k}(s))\Upsilon_1(r_3,\rho,\lambda_{z,5,n})d\rho \\
g_{7,8,n}(s) = \frac{4}{\pi r_1 h_d} \sum_{k=0}^{+\infty} \tilde{R}_{7,k}(\hat{I}_e(s))\frac{1}{\Lambda_{r,8,k}(s)\cosh\left(h_d\Lambda_{r,8,k}(s)\right)} \int_{z_{-1}}^{z_{-2}} \sinh\left((\xi-z_{-2})\Lambda_{r,8,k}(s)\right) \sin\left((\xi-z_{-2})\sqrt{\lambda_{r,8,n}}\right) d\xi \\
g_{1,8,n}(s) = \frac{1}{\nu_{1n}} \sum_{k=0}^{+\infty} \frac{(-1)^k \tilde{Z}_{1,k}(\hat{I}_e(s))}{\Lambda_{r,1,k}(s)\Xi_0(r_0,r_1,\Lambda_{r,1,k}(s))} \int_{r_0}^{r_1} \rho \Xi_1(r_0,\rho,\Lambda_{r,1,k}(s))\Upsilon_1(r_0,\rho,\lambda_{z,1,n})d\rho \\
h_{1,2,k,n}(s) = \frac{(-1)^k}{\nu_{1n}} \int_{r_0}^{r_1} \rho \Xi_1(r_0,\rho,\Lambda_{r,2,k}(s))\Upsilon_1(r_0,\rho,\lambda_{z,1,n})d\rho \\
h_{2,3,k,n}(s) = \frac{2}{z_2-z_1}\Upsilon_1(r_0,r_1,\lambda_{z,1,k}) \int_{z_1}^{z_2} \sinh\left((z_2-\xi)\Lambda_{z,1,k}(s)\right) \sin\left((z_2-\xi)\sqrt{\lambda_{r,2,n}}\right) d\xi \\
h_{3,4,k,n}(s) = \frac{2}{z_2-z_1}\Upsilon_1(r_3,r_2,\lambda_{z,4,k}) \int_{z_1}^{z_2} \sinh\left((z_2-\xi)\Lambda_{z,4,k}(s)\right) \sin\left((z_2-\xi)\sqrt{\lambda_{r,3,n}}\right) d\xi \\
h_{4,5,k,n}(s) = \frac{(-1)^k}{\nu_{2n}} \int_{r_2}^{r_3} \rho \Xi_1(r_3,\rho,\Lambda_{r,4,k}(s))\Upsilon_1(r_3,\rho,\lambda_{z,5,n})d\rho \\
h_{6,7,k,n}(s) = \frac{2}{h_d}\Upsilon_1(r_3,r_2,\lambda_{z,6,k}) \int_{z_{-1}}^{z_{-2}} \sinh\left((\xi-z_{-2})\Lambda_{z,6,k}(s)\right) \sin\left((\xi-z_{-2})\sqrt{\lambda_{r,7,n}}\right) d\xi \\
h_{7,8,k,n}(s) = \frac{2}{h_d}\Upsilon_1(r_0,r_1,\lambda_{z,8,k}) \int_{z_{-1}}^{z_{-2}} \sinh\left((\xi-z_{-2})\Lambda_{z,8,k}(s)\right) \sin\left((\xi-z_{-2})\sqrt{\lambda_{r,7,n}}\right) d\xi
\end{array}
\tag{38}
$$

$\overline{\mathscr{A}}(s)$ is thus obtained by combining (36),(37) and (33). Its expression, very long and complicated, is omitted due to the lack of space.

A.2 Expression of the effective flux and the induced flux in the MTB

We call effective flux, Φ, the magnetic flux created in the MTB, taking the eddy currents effects into account. This flux can be seen as the difference of two fluxes: the flux created in the same MTB without eddy currents, Φ_T, and the induced flux created by eddy currents, Φ_e:

$$|\Phi(t)| = |\Phi_T(t) - \Phi_e(t)|$$

We define the effective flux in the air gap, in frequency domain, as the flux at the boundary between Ω_1 and the air gap:

$$
\begin{aligned}
\hat{\Phi}(s) &= \iint \vec{B}_1 d\vec{S}_1 \\
&= \oint \vec{A}_1 d\vec{l}_1 \quad \text{from Stokes' theorem} \\
&= 2\pi\left(r_1\hat{A}_1(r_1,z_0,s) - r_0\hat{A}_1(r_0,z_0,s)\right) \\
&= 2\pi r_1 \hat{I}(s)\left(\frac{2}{z_1-z_0}\sum_{k=0}^{+\infty}\tilde{Z}_{1,k}(\hat{I}_e(s))\frac{\Xi_1(r_0,r_1,\Lambda_{r,1,k}(s))}{\Lambda_{r,1,k}(s)\Xi_0(r_0,r_1,\Lambda_{r,1,k}(s))} + \sum_{k=0}^{+\infty} b'_{1,k}(s)\Upsilon_1(r_0,r_1,\lambda_{z,1,k})\right)
\end{aligned}
$$

Combining this expression with (33), we get:

$$\hat{\Phi}(s) = \left(\frac{2}{z_1 - z_0} \sum_{k=0}^{+\infty} \tilde{Z}_{1,k}(\hat{l}_e(s)) \frac{\Xi_1(r_0, r_1, \Lambda_{r,1,k}(s))}{\Lambda_{r,1,k}(s)\Xi_0(r_0, r_1, \Lambda_{r,1,k}(s))} + \sum_{k=0}^{+\infty} b'_{1,k}(s)\Upsilon_1(r_0, r_1, \lambda_{z,1,k}) \right) \frac{2\pi r_1 \hat{y}(s)}{\hat{G}(s)} \tag{39}$$

Eddy currents vanishing at steady state, the flux created in the MTB without eddy currents may be obtained if $y(t)$ is at steady state at each time t. Using (39), Φ_T reads:

$$\Phi_T(s) = 2\pi r_1 \left(\frac{2}{z_1 - z_0} \sum_{k=0}^{+\infty} \tilde{Z}_{1,k}(\hat{l}_e(s)) \frac{\Xi_1(r_0, r_1, \Lambda_{r,1,k}(0))}{\Lambda_{r,1,k}(0)\Xi_0(r_0, r_1, \Lambda_{r,1,k}(0))} + \sum_{k=0}^{+\infty} b'_{1,k}(0)\Upsilon_1(r_0, r_1, \lambda_{z,1,k}) \right) \frac{\hat{y}(s)}{\hat{G}(0)} \tag{40}$$

Finally, we get Φ_e by combining (39) and (40) with:

$$|\Phi_e(t)| = |\Phi_T(t) - \Phi(t)|$$

In frequency domain, we define the function $\mathscr{F} : \sqrt{-1}\,\mathbb{R} \to \mathbb{C}$ such that $\hat{\Phi}_e(s) = \overline{\mathscr{F}_e}(s)y(s)$:

$$\begin{aligned} \overline{\mathscr{F}_e}(s) = \frac{4\pi r_1}{(z_1 - z_0)G(s)} & \left(\sum_{k=0}^{+\infty} \left(\tilde{Z}_{1,k}(\hat{l}_e(s)) \frac{\Xi_1(r_0, r_1, \Lambda_{r,1,k}(0))}{\Lambda_{r,1,k}(0)\Xi_0(r_0, r_1, \Lambda_{r,1,k}(0))} \right. \right. \\ & \left. - \tilde{Z}_{1,k}(\hat{l}_e(s)) \frac{\Xi_1(r_0, r_1, \Lambda_{r,1,k}(s))}{\Lambda_{r,1,k}(s)\Xi_0(r_0, r_1, \Lambda_{r,1,k}(s))} \right) \\ & \left. + \sum_{k=0}^{+\infty} \left(b'_{1,k}(0)\Upsilon_1(r_0, r_1, \lambda_{z,1,k}) - b'_{1,k}(s)\Upsilon_1(r_0, r_1, \lambda_{z,1,k}) \right) \right) \end{aligned} \tag{41}$$