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ROBUST CONTROL AGAINST DISTURBANCE MODEL UNCERTAINTY IN ACTIVE MAGNETIC BEARINGS

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ABSTRACT

Synchronous vibrations in rotors suspended by active magnetic bearings are caused by an unbalanced mass. These are characterized as sinusoidal disturbances with a frequency that is equal to the rotational speed. In the past, robust controllers based on H_∞ loop shaping design procedure (LSDP) and Q-parameterization theory have been employed to reject sinusoidal disturbances of a particular frequency. Variations or inaccurate measurements of the rotor operating speed, as seen in real life, motivate us to treat the frequency of disturbance as an uncertain parameter. In this paper, a novel robust controller design procedure is applied to an active magnetic bearing system.

INTRODUCTION

In recent years, active magnetic bearings (AMBs) are used in a number of industrial applications, viz., hard disk drives, flywheel energy storage systems and milling applications. In these applications, it is highly desirable to reject sinusoidal disturbances which occur due to mass imbalances. From [7], it is known that the frequency of this periodic disturbance is equal to the rotational speed of the spindle. In applications such as micro-milling, the rotational speed of the spindle could deviate by a small percentage from its intended operating speed (owing to tool-workpiece interaction), which correspond to variations in the frequency of sinusoidal disturbance affecting the plant. Hence, we regard this variation as an uncertainty and address the problem of robust control design against disturbance model uncertainty.

Rejecting sinusoidal disturbances in magnetic bearings is also known as unbalance compensation. A commonly used approach to design robust controllers is based on the assumption that the rotational speed is constant. An analytical model for a magnetic bearing spindle incorporating the influence of mass imbalance was first given by Matusumura et al in [7] and [8]. Two different models were presented that describe the rotation about a geometric axis and the so-called inertia-axis. The same authors successfully applied an H_∞ loop shaping design procedure (LSDP) to robustly stabilize a magnetic bearing system against uncertainties in the plant model in [4]. The LSDP method was extended to reject sinusoidal disturbances of a particular frequency in [5]. The design was aimed at rejecting disturbances of a particular frequency, in order to ensure rotation about a fixed geometric axis.

Mohemed et. al. proposed the use of Q-parametrization technique for imbalance compensation in [10]. Here, a simplified system model is considered in order to achieve spindle rotation about the inertia axis and geometric axis respectively. The approaches presented in [4] and [10] rely on choosing a particular controller from the set of parameterized sub-optimal H_∞ controllers. The essential design philosophy is to have a controller with a pole on the imaginary axis at the desired frequency. Apart from H_∞ based controllers, use of other techniques based on using a notch filter in closed loop and from adaptive control literature are given in [6] and [15] respectively.

The robust control design problem for an LTI system with uncertainty is generally understood to be non-convex. In a recent publication, Dietz et. al. [2] have considered the robust synthesis problem when the uncertainty only affects a disturbance filter at the plant input. Based on a suitable combination of transformations found in the literature, a solution to the related synthesis problem was given as a set of Linear Matrix Inequalities (LMIs).

In this paper, we consider the application of this control design procedure to an active magnetic bearing system. The class of sinusoidal disturbance signals, with time-varying frequency, can be effectively modeled as the output of an uncertain filter, excited by an impulse. The plant is assumed to be an LTI system.

MODELING

A horizontal-spindle suspended by two radial active magnetic bearings is shown in Fig. 1. Each AMB consists of four electro-magnets with two in the vertical and two in the horizontal plane. The electro-magnets on the left side and in the vertical plane are named as l_1, l_2. The ones in the horizontal plane are named as l_3, l_4. A similar naming scheme is used for the electro-magnets on the right side of Figure 1.

Each pair of opposite electro-magnets operates in a differential mode. That is, if I denotes the equilibrium current, an incremental change leads to currents $I + i$ and $I - i$ in opposite electro-magnets. The rigid body dynamics of such a magnetic suspension system can be found in [7] and [8].

The axis system X_G-Z_G lies at the geometric center of the spindle and X_I-Z_I lies at the inertia center. The radial offset in the position of O_G and O_I is attributed to the presence of unbalanced mass. For micro-milling, it is required to ensure rotation about a fixed geometric axis.

Rotation about geometric axis

It is desired to regulate the deviations in the gap lengths between the electro-magnets and the $X_G - Z_G$ axis system. For brevity, the vertical and horizontal variables are grouped together as

$$\begin{aligned} x_v &= \begin{bmatrix} g_{r1} & g_{l1} & \dot{g}_{r1} & \dot{g}_{l1} & i_{r1} & i_{l1} \end{bmatrix}^T \\ x_h &= \begin{bmatrix} g_{r3} & g_{l3} & \dot{g}_{r3} & \dot{g}_{l3} & i_{r3} & i_{l3} \end{bmatrix}^T \\ u_v &= \begin{bmatrix} e_{r1} & e_{l1} \end{bmatrix}^T \\ u_h &= \begin{bmatrix} e_{r3} & e_{l3} \end{bmatrix}^T . \end{aligned} \quad (1)$$

The variables g and e denote the deviations in gap length and the supplied voltage respectively at the subscripted electro-magnet. As shown in [7, 8], considering small deviations from the equilibrium position of spindle, the following state-space equations can be derived:

$$\begin{aligned} \begin{bmatrix} \dot{x}_v \\ \dot{x}_h \end{bmatrix} &= \begin{bmatrix} A_v & \omega A_{vh} \\ \omega A_{hv} & A_h \end{bmatrix} \begin{bmatrix} x_v \\ x_h \end{bmatrix} + \begin{bmatrix} B_v & 0 \\ 0 & B_h \end{bmatrix} \begin{bmatrix} u_v \\ u_h \end{bmatrix} \\ &\quad + \omega^2 \begin{bmatrix} E_v & 0 \\ 0 & E_h \end{bmatrix} v \\ y &= \begin{bmatrix} C_v & 0 \\ 0 & C_h \end{bmatrix} \begin{bmatrix} x_v \\ x_h \end{bmatrix} . \end{aligned} \quad (2)$$

Here, v denotes the disturbance and y the measured deviations in the gap lengths, i.e. $y = \begin{bmatrix} g_{r1} & g_{l1} & g_{r3} & g_{l3} \end{bmatrix}^T$. Note that the disturbance v is amplified by the square of the rotational speed of the spindle. Further, this disturbance is sinusoidal and characterized by the mass imbalance related parameters ϵ, τ and initial offsets μ, λ as

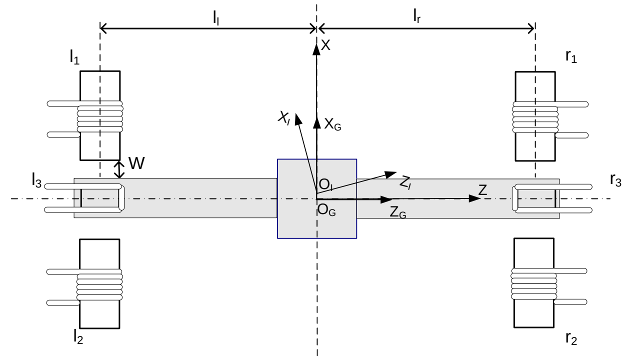

Figure 1: HORIZONTAL SPINDLE IN MAGNETIC BEARINGS

$$v = \begin{bmatrix} \epsilon \cos(\omega t + \mu) \\ \tau \cos(\omega t + \lambda) \\ \epsilon \sin(\omega t + \mu) \\ \tau \sin(\omega t + \lambda) \end{bmatrix} . \quad (3)$$

PROBLEM FORMULATION

Consider the problem of designing a robust $\mathcal{H}_\infty$-optimal controller for the interconnection of Figure 2, in order to obtain a guaranteed performance level for all disturbance signals that are given by an uncertain filter. Let the following minimal realization of the linear time-invariant (LTI) plant P be given:

$$P := \left[\begin{array}{c|cc} A & B_v & B_u \\ \hline C_z & D_{zv} & D_{zu} \\ C_y & D_{yv} & 0 \end{array} \right],$$

in which $A \in \mathbb{R}^{n \times n}$. Assume that the disturbance filter W is perturbed by an uncertain element Δ that is allowed to be any element in a given set $\boldsymbol{\Delta}$. The dependence on Δ is modelled by a linear fractional transformation, written as $\Delta \star W = W_{vw} + W_{vp}\Delta(I - W_{qp}\Delta)^{-1}W_{qw}$, in which the nominal filter W can be represented as

$$W := \begin{pmatrix} W_{qp} & W_{qw} \\ W_{vp} & W_{vw} \end{pmatrix} = \left[\begin{array}{c|cc} A_W & B_p & B_w \\ \hline C_q & D_{qp} & D_{qw} \\ C_v & D_{vp} & D_{vw} \end{array} \right]$$

where $A_W \in \mathbb{R}^{n_W \times n_W}$. All eigenvalues of A_W are assumed to lie in the left half plane since the controller is unable to influence the dynamics of W. The uncertain element Δ can consist of various types of norm-bounded non-linearities or dynamic time-varying operators.

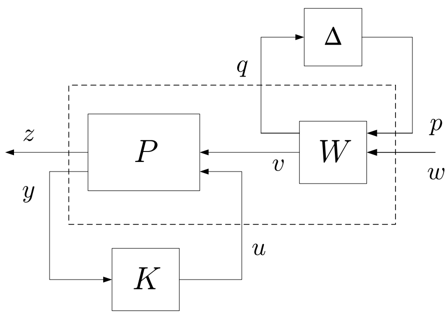

Figure 2: Systems interconnection with uncertain filter W

Problem Statement
For a given filter W and set of uncertainties $\mathbf{\Delta}$, design an LTI controller denoted by

$$K := \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \tag{4}$$

that, once interconnected with P, leads to a guaranteed robust $\mathcal{H}_\infty$-performance level from w to z.

SYNTHESIS PROCEDURE

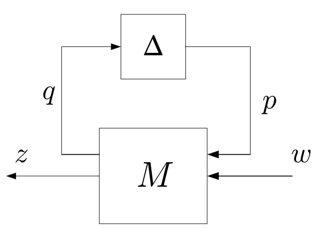

Figure 3: Standard interconnection for robust performance analysis

In the sequel, we will work with the so called IQC (Integral Quadratic Constraints) characterization of the uncertainty block. It belongs to a set $\mathbf{\Delta}$, which in-turn is described by a structured matrix Π, called the multiplier. The set $\mathbf{\Delta}$ contains 0 as an element. The input-output channels of the uncertainty block satisfy the following quadratic constraint.

$$\left\langle \begin{pmatrix} \Delta q \\ q \end{pmatrix}, \Pi \begin{pmatrix} \Delta q \\ q \end{pmatrix} \right\rangle_{\mathcal{L}_{2+}} \succeq 0, \quad \forall q \in \mathcal{L}_{2+}, \forall \Delta \in \mathbf{\Delta}. \tag{5}$$

The matrix $\Pi \in \mathbf{\Pi}$ is assumed to be a static multiplier. The set $\mathbf{\Pi}$ allows one to capture various types of non-linearities or time-varying operators, [9, 3]. In particular, for a single, possibly time-varying parameter $\delta \in [-1, 1]$, repeated such that $\Delta = \delta I$, condition (5) holds for all elements Π of

$$\mathbf{\Pi}_{DG} : \left\{ \begin{pmatrix} -D & G \\ G' & D \end{pmatrix} : \qquad D \succ 0, \quad G = -G' \right\} \tag{6}$$

For the following theorem, let us be given a stable transfer matrix M with the realization

$$M := \begin{pmatrix} M_{qp} & M_{qw} \\ M_{zp} & M_{zw} \end{pmatrix} = \left[\begin{array}{c|cc} A & B_p & B_w \\ \hline C_q & D_{qp} & D_{qw} \\ C_z & D_{zp} & D_{zw} \end{array}\right]. \tag{7}$$

Theorem 1 *For a given system M in Figure 3, we have a robust $\mathcal{H}_\infty$-performance level smaller than γ, if for some $\Pi \in \mathbf{\Pi}$ there exists $\mathcal{X} \succ 0$ such that*

$$(\ ..\ )' \left(\begin{array}{cc|c|c} 0 & \mathcal{X} & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 \\ \hline 0 & 0 & \Pi & 0 \\ \hline 0 & 0 & 0 & J \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ A & B_p & B_w \\ \hline 0 & I & 0 \\ C_q & D_{qp} & D_{qw} \\ \hline 0 & 0 & I \\ C_z & D_{zp} & D_{zw} \end{array}\right) \prec 0 \tag{8}$$

where

$$J = \begin{pmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma} I \end{pmatrix}. \tag{9}$$

Proof: see [1, 13].

In nominal $\mathcal{H}_\infty$-synthesis, Theorem 1 is considered for the closed loop system matrices and applying a linearizing variable transformation (see [13]) removes the bilinearity between controller parameters and Lyapunov matrix $\mathcal{X}$. In case the system dynamics are uncertain, it is not known how to render the synthesis problem tractable. Hence, usually controller/scalings iterations are employed to obtain solutions. We will show that for the particular class of systems as introduced in the previous section, no such iterations are required.

Robust $\mathcal{H}_\infty$-synthesis with static scalings
In order to apply the robust $\mathcal{H}_\infty$-norm characterization (8) in Theorem 1 we put the problem in the standard robust control synthesis framework by merging the dynamics of W with the plant P to obtain the following realization of the extended plant:

$$\left[\begin{array}{cc|ccc} A & B_v C_v & B_v D_{vp} & B_v D_{vw} & B_u \\ 0 & A_W & B_p & B_w & 0 \\ \hline 0 & 0 & I & 0 & 0 \\ 0 & C_q & D_{qp} & D_{qw} & 0 \\ \hdashline 0 & 0 & 0 & I & 0 \\ C_z & D_{zv} C_v & D_{zv} D_{vp} & D_{zv} D_{vw} & D_{zu} \\ \hdashline C_y & D_{yv} C_v & D_{yv} D_{vp} & D_{yv} D_{vw} & 0 \end{array}\right]. \tag{10}$$

$$\left(\begin{array}{c}\mathbf{A}\\ \hline \mathbf{C}_z\end{array}\right)=\left(\begin{array}{cccc} AT_{11}+B_u\bar{M}_1 & -AT_{12}+B_vC_v+T_{12}A_W+B_u\bar{M}_2 & A+B_uNC_y & B_vC_v+T_{12}A_W+B_uND_{yv}C_v\\ 0 & T_{22}A_W & 0 & T_{22}A_W\\ \bar{K}_1 & \bar{K}_2 & \begin{array}{c}X_{11}A+L_1C_y\\ X_{21}A+L_2C_y\end{array} & \begin{array}{c}X_{11}B_vC_v+X_{12}A_W+L_1D_{yv}C_v\\ X_{21}B_vC_v+X_{22}A_W+L_2D_{yv}C_v\end{array}\\ \hline C_zT_{11}+D_{zu}\bar{M}_1 & -C_zT_{12}+D_{zv}C_v+D_{zu}\bar{M}_2 & C_z+D_{zu}NC_y & D_{zv}C_v+D_{zu}ND_{yv}C_v \end{array}\right) \tag{*}$$

As usual we assume that $D_{yu}=0$ without loss of generality.

Theorem 2 *Suppose we are given the interconnection in Figure 2, the uncertainty set* $\mathbf{\Delta}$ *and scalings* $\Pi\in\mathbf{\Pi}$ *satisfying (5). Let matrices* $T, X, \bar{K}, L, \bar{M}$ *be partitioned as*

$$T=\begin{pmatrix} T_{11} & T_{12}\\ T_{12}' & T_{22}\end{pmatrix},\qquad X=\begin{pmatrix} X_{11} & X_{12}\\ X_{12}' & X_{22}\end{pmatrix} \tag{11}$$

and

$$\begin{pmatrix}\bar{M}\\ \bar{K}\end{pmatrix}=\begin{pmatrix}\bar{M}_1 & \bar{M}_2\\ \bar{K}_1 & \bar{K}_2\end{pmatrix},\qquad L=\begin{pmatrix}L_1\\ L_2\end{pmatrix}$$

in which $T_{11}, X_{11}, \bar{K}_1, \bar{M}_1, L_1'$ *and* $T_{22}, X_{22}, \bar{K}_2, \bar{M}_2, L_2'$ *have* n *and* n_W *columns respectively. Then, there exists a controller such that the robust* $\mathcal{H}_\infty$*-norm from* $w\to z$ *is at most* γ *if there exists* $\{T, X, \bar{K}, L, \bar{M}, N\}$ *and* $\Pi\in\mathbf{\Pi}$ *for which*

$$(\;..\;)'\left(\begin{array}{cc|c|c} 0 & I & 0 & 0\\ I & 0 & 0 & 0\\ \hline 0 & 0 & \Pi & 0\\ \hline 0 & 0 & 0 & J\end{array}\right)\left(\begin{array}{ccc} I & 0 & 0\\ \mathbf{A} & \mathbf{B}_p & \mathbf{B}_w\\ \hline 0 & I & 0\\ C_e & D_{qp} & D_{qw}\\ \hline 0 & 0 & I\\ \mathbf{C}_z & \mathbf{D}_{zp} & \mathbf{D}_{zw}\end{array}\right)\prec 0, \tag{12}$$

$$\left(\begin{array}{cc|cc} T_{11} & 0 & I & T_{12}\\ 0 & T_{22} & 0 & T_{22}\\ \hline I & 0 & X_{11} & X_{12}\\ T_{12}' & T_{22} & X_{21} & X_{22}\end{array}\right)\succ 0, \tag{13}$$

where $\mathbf{A}, \mathbf{C}_z$ *are given in (*) at the top of the page,* J *is given by (9) and*

$$\begin{aligned}
\mathbf{B}_p &= \begin{pmatrix} B_vD_{vp}+T_{12}B_p+B_uND_{yp}\\ T_{22}B_p\\ X_{11}B_vD_{vp}+X_{12}B_p+L_1D_{yp}\\ X_{21}B_vD_{vp}+X_{22}B_p+L_2D_{yp}\end{pmatrix},\\
\mathbf{B}_w &= \begin{pmatrix} B_vD_{vw}+T_{12}B_w+B_uND_{yw}\\ T_{22}B_w\\ X_{11}B_vD_{vw}+X_{12}B_w+L_1D_{yw}\\ X_{21}B_vD_{vw}+X_{22}B_w+L_2D_{yw}\end{pmatrix},\\
\mathbf{D}_{zp} &= D_{zv}D_{vp}+D_{zu}ND_{yv}D_{vp},\\
\mathbf{D}_{zw} &= D_{zv}D_{vw}+D_{zu}ND_{yv}D_{vw},\\
C_e &= \begin{pmatrix} 0 & C_q & 0 & C_q\end{pmatrix}.
\end{aligned} \tag{14}$$

Note that all boldface symbols depend on the decision variables in an affine fashion.

Moreover, an application of the Schur complement formula renders condition (12) affine in $\{\gamma, \mathbf{C}_z, \mathbf{D}_{zp}, \mathbf{D}_{zw}\}$ *which allows to infimize* γ *and compute sub-optimal controllers.*

Proof: The result is obtained by combining two transformations taken from the existing literature. The first variable transformation is taken from [13] and essentially convexifies the bilinearity between the closed loop system matrices and the Lyapunov matrix as present in (8). For notational reasons, we introduce the following abbreviation for the extended plant (10):

$$\left[\begin{array}{c|ccc} \tilde{A} & \tilde{B}_p & \tilde{B}_w & \tilde{B}\\ \hline 0 & I & 0 & 0\\ \tilde{C}_q & D_{qp} & D_{qw} & 0\\ \hdashline 0 & 0 & I & 0\\ \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} & \tilde{D}_{zu}\\ \hdashline \tilde{C} & D_{yp} & D_{yw} & 0\end{array}\right]. \tag{15}$$

In fact, the robust $\mathcal{H}_\infty$-synthesis conditions in [13] amount to the existence of variables $\{X, Y, K, L, M, N\}$ and $\Pi\in\mathbf{\Pi}$ for which

$$\begin{pmatrix} Y & I\\ I & X\end{pmatrix}\succ 0, \tag{16}$$

and (12) holds with the substitution

$$\begin{aligned}
\mathbf{A} &\to \begin{pmatrix} \tilde{A}Y+\tilde{B}M & \tilde{A}+\tilde{B}N\tilde{C}\\ K & X\tilde{A}+L\tilde{C}\end{pmatrix},\\
\mathbf{B}_p &\to \begin{pmatrix} \tilde{B}_p+\tilde{B}ND_{yp}\\ X\tilde{B}_p+LD_{yp}\end{pmatrix},\\
\mathbf{B}_w &\to \begin{pmatrix} \tilde{B}_w+\tilde{B}ND_{yw}\\ X\tilde{B}_w+LD_{yw}\end{pmatrix},\\
\mathbf{C}_z &\to \begin{pmatrix} \tilde{C}_zY+\tilde{D}_{zu}M & \tilde{C}_z+\tilde{D}_{zu}N\tilde{C}\end{pmatrix},\\
\mathbf{D}_{zp} &\to \tilde{D}_{zp}+\tilde{D}_{zu}ND_{yp},\\
\mathbf{D}_{zw} &\to \tilde{D}_{zw}+\tilde{D}_{zu}ND_{yw},\\
C_e &\to \mathbf{C}_q=\begin{pmatrix} \tilde{C}_qY & \tilde{C}_q\end{pmatrix},
\end{aligned} \tag{17}$$

corresponding to the realization (10). The coupling condition (16) is related to positivity of the Lyapunov matrix and therefore implies nominal stability of the closed loop system. Condition (12) still remains non-convex due to bilinearity between scalings Π and $\mathbf{C}_q = (\tilde{C}_q Y \quad \tilde{C}_q)$. By using the particular structure seen in realization matrices (10) we can overcome this trouble. In order to do so, with Y partitioned similar as T in (11), the mapping $Y \to T$, defined by (see [12, 11])

$$T_{22} = Y_{22}^{-1}, \quad T_{12} = -Y_{12}Y_{22}^{-1}, \quad T_{11} = Y_{11} + Y_{12}T_{12}'$$

is bijective on the space of symmetric matrices.

Next, we apply a congruence transformation on (12), when substitutions have been performed as in (17), by left- and right-multiplication of the first row/column with

$$\begin{pmatrix} I & T_{12} & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & 0 & 0 \\ T_{12}' & T_{22} & 0 \\ 0 & 0 & I \end{pmatrix} \tag{18}$$

respectively. What remains is condition (12) with substitutions performed as indicated in (14). Similarly, we left- and right multiply the coupling condition (16) with the first and second term in (18) respectively, resulting in (13). Since C_e in (14) now no longer depends on the decision variable Y, condition (12) is in fact affine in the variables $X, T, \bar{K}, L, \bar{M}, N$ and scalings Π.

The controller reconstruction formulae are taken from [13], where further details can be found. In order to reconstruct the controller matrices, let

$$Y = \begin{pmatrix} T_{11} + T_{12}T_{22}^{-1}T_{12}' & -T_{12}T_{22}^{-1} \\ -T_{22}^{-1}T_{12}' & T_{22}^{-1} \end{pmatrix} \tag{19}$$

and find matrices U, V such that $UV^T = I - XY$. Then with

$$\begin{pmatrix} \hat{K} \\ \hat{M} \end{pmatrix} = \begin{pmatrix} \bar{K} \\ \bar{M} \end{pmatrix} \begin{pmatrix} I & 0 \\ T_{12}' & T_{22} \end{pmatrix}^{-1}$$

the controller matrices can be obtained as

$$\begin{aligned} D_K &:= N, \\ C_K &:= (\hat{M} - D_K\tilde{C}X)U^{-T}, \\ B_K &:= V^{-1}(L - Y\tilde{B}D_K), \\ A_K &:= V^{-1}\big(\hat{K} - VB_K\tilde{C}X \\ &\quad -Y\tilde{B}C_KU^T - Y(\tilde{A} + \tilde{B}D_K\tilde{C})X\big)U^{-T}. \end{aligned} \tag{20}$$

Remark 3 *We strongly emphasize that no assumption was needed on the structure of the multiplier $\Pi \in \mathbf{\Pi}$. This means that robust synthesis is possible for any set of multipliers satisfying (5). In particular, suitable multiplier classes for multi-parameter regions described by polynomial inequalities can be constructed by relaxing condition (5) using sum-of-squares techniques. For an overview on different relaxation schemes, see [14].*

Table 1: MODEL PARAMETERS

Parameters of the model	(SI units)
m	13.9 (kg)
J_z	0.01348 (kg/m^2)
J_x	0.2326 (kg/m^2)
l_l, l_r	0.13 (m)
f_{l_1,r_1}	90.9 (N)
$f_{l_2\sim l_4}, f_{r_2\sim r_4}$	22 (N)
I_{l_1,r_1}	0.63 (A)
$I_{l_2\sim l_4}, I_{r_2\sim r_4}$	22 (N)
W	5.5×10^{-4} (m)

Remark 4 *The computational complexity of the system of LMIs could be reduced by eliminating the controller parametrs in the the robust $\mathcal{H}_\infty$-synthesis. The interested reader is referred to [2] for further details.*

CONTROL OF THE AMB SYSTEM

We now apply the control design procedure to an AMB system with model parameters given in Table 1, [4]. Our primary interest for the closed loop system, is the suppression of sinusoidal disturbances. Hence, we adopt an S/KS design procedure based on the interconnection of Figure 4. When the uncertainty is time invariant, the corresponding optimization problem is to: Find an LTI controller K that minimizes the value of γ such that

$$\max_{\delta\in[-r,r]} \left\| \begin{matrix} W_p(I + G_0K)^{-1}(\delta I * W) \\ W_uK(I + G_0K)^{-1}(\delta I * W) \end{matrix} \right\|_\infty < \gamma. \tag{21}$$

However, we regard the uncertainty as time-varying uncertainty and minimize the $\mathcal{L}_2$ gain of the closed loop system. The plant G_0 is an LTI system corresponding to a fixed rotational speed of 1200 rpm (125.6637 [rad/s]). The weighting functions on the sensitivity and the control input are choosen as

$$W_p = \frac{10(0.5s + 2)}{(s + 2 \times 10^{-2})} \begin{bmatrix} 200 & 0 & 0 & 0 \\ 0 & 200 & 0 & 0 \\ 0 & 0 & 350 & 0 \\ 0 & 0 & 0 & 350 \end{bmatrix}, \tag{22}$$

$$W_u = \frac{10^{-4}(s + 100)}{(10^{-5}s + 100)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

The disturbance v entering the plant interconnection is a 4-channel input in accordance with (3). Recall that a sinusoidal signal of frequency $\omega_0(1 + \delta)$ can be modeled

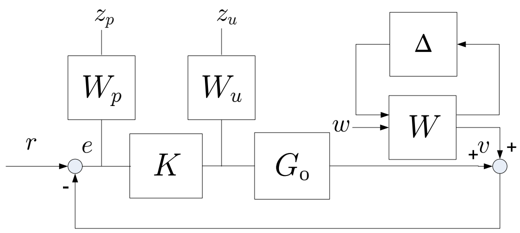

Figure 4: Interconnection of example.

as the output of the autonomous system

$$\dot{\xi} = \begin{pmatrix} 0 & \omega_0(1+\delta) \\ -\omega_0(1+\delta) & 0 \end{pmatrix} \xi, \quad \xi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{24}$$

in which δ is the time-varying parameter bounded by r.

Since the algorithm outlined in the previous section requires the filter to be stable, we slightly perturb the system matrix in (24) by adding a non-zero damping term. To be precise, the following realization is chosen to represent our disturbances.

$$\begin{aligned} \dot{\xi} &= \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{pmatrix} \xi + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w \\ &\quad + \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{pmatrix} p \\ q &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xi \\ v &= \begin{pmatrix} 0 & 2\omega_0 \end{pmatrix} \xi + \begin{pmatrix} 0 & 2\omega_0 \end{pmatrix} p + \kappa w, \end{aligned} \tag{25}$$

and the parameter trajectory $\delta(t)$ satisfies $\delta(t) \in [-r, r]$ for all t. Closing the uncertainty channel amounts to setting $p = \delta(t)q$. If the parameters are time-invariant, i.e. $\delta(t) = \delta$ for some $\delta \in [-r, r]$, the dynamical system (27) corresponds to the uncertain filter

$$F_\delta(s, \delta) = \delta I \star F(s) = \kappa + \frac{2\zeta\omega_\delta s}{s^2 + 2\zeta\omega_\delta s + \omega_\delta^2}, \tag{26}$$

in which $\omega_\delta = \omega_0(1+\delta)$. Hence, the system matrices of the filter W are given as

$$\left(\begin{array}{c|c|c} A_W & B_p & B_w \\ \hline C_q & D_{qp} & D_{qw} \\ \hline C_v & D_{vp} & D_{vw} \end{array}\right) = \left(\begin{array}{cc|cc|c} 0 & \omega_0 & 0 & \omega_0 & 1 \\ -\omega_0 & -2\zeta\omega_0 & -\omega_0 & -2\zeta\omega_0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 2\omega_0 & 0 & 2\omega_0 & \epsilon \end{array}\right). \tag{27}$$

Discussion of results

We first compute a nominal controller without considering an uncertainty in the weighting filter by minimizing the H_∞ norm from w to the output channels, using the LMI based solver in MATLAB (*hinfsyn*). This design is then compared to the controllers obtained by the procedure described in this paper for $r \in \{0.005, 0.01, 0.02\}$. The damping for all the designs is taken to be $\zeta = 10^{-5}$. Moreover, we set $\kappa = 0.01$, which is merely a tuning variable.

Figure 5 shows the frequency response of the closed loop sensitivity for the nominal design as well as the three robust designs. We observe that the notch (signifying disturbance rejection) is wider for the robust designs. The low frequency gain is reduced at the expense of an increased gain in the range 300-1000 rad/s.

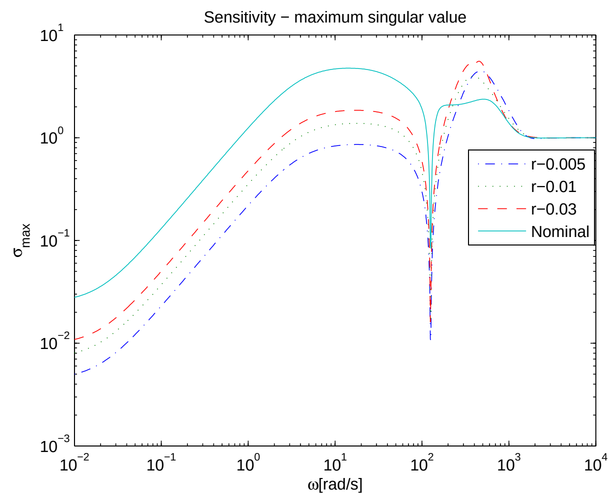

Figure 5: Frequency response of $(I + GK)^{-1}$.

For a detailed analysis, consider the zoomed in plot of the sensitivity in Figure 6. For most frequencies in the region of interest $[\omega_0(1-r), \omega_0(1+r)]$, the robust designs yield better disturbance attenuation than the nominal design. However, at the nominal frequency $\omega_0 = 1200$ rpm, the robust design corresponding to $r = 0.03$ performs worse, a fact which also holds for the other two robust designs. This is due to the fact that the robust synthesis algorithm aims at reducing the gain over a wider frequency range, which inevitably results in a less sharp notch.

When the three robust designs are compared to one another, it should be observed that increasing the parameter bound from 0.005 to 0.03 does not lead to better attenuation in the frequency region of interest. In other words, we do not see any trade-off in the performance for different parameter bounds. Moreover, the deterioration in performance as the parameter bound r grows is also seen by the increased values of γ, going from 3 to 10 to 30 for the parameter bounds 0.005, 0.01 and 0.03 respectively. This is clearly seen as an increased gain in the sensitivity plots.

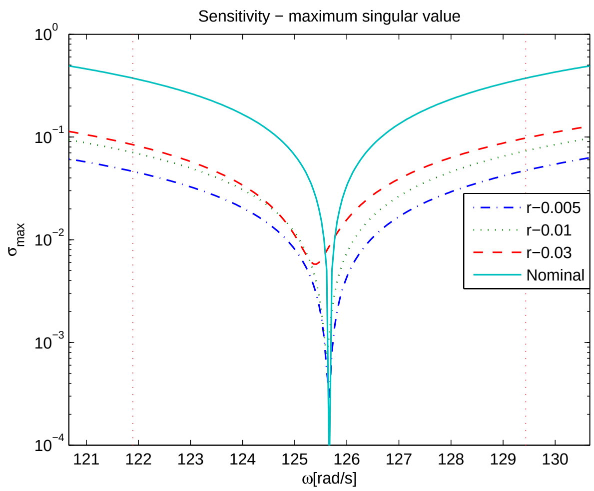

Figure 6: Zoom in: Frequency response of $(I + GK)^{-1}$.

This observation is probably due to the fact that we allow the uncertain parameter be time-varying. Further, if using constant multiplier matrices Π in the controller synthesis algorithm, we are actually guaranteeing robust performance against arbitrary parameter variations. This source of conservatism could possibly be circumvented by considering frequency dependent scalings for the multiplier matrices, which will be investigated in an upcoming paper.

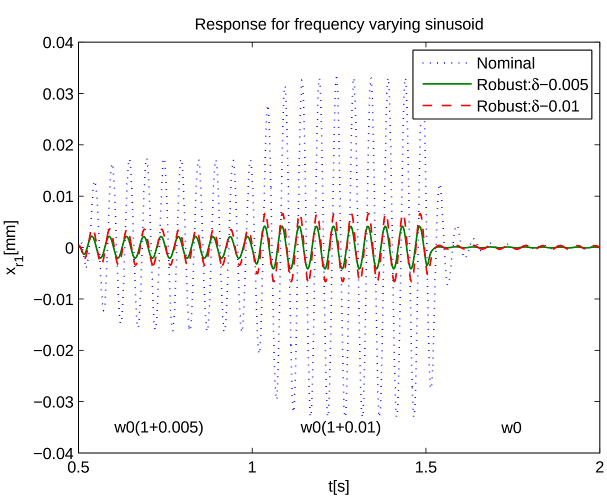

Figure 7: Time domain response

Figure 7 shows the closed loop response for nominal and robust designs (displacements at location r_1), subject to a time-varying sinusoidal disturbance. The sinusoidal signal is generated by the undamped system (24), with imbalance parameters $\epsilon = 0.01$ mm, $\tau = 0.01$ rad. Clearly, the robust designs result in better attenuation levels if the frequency varies with time.

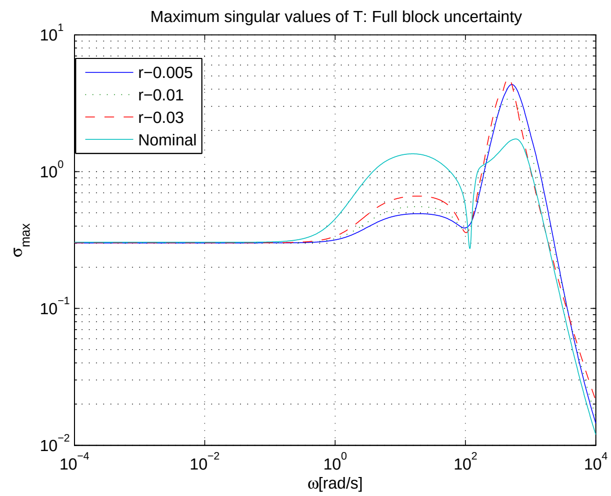

Figure 8: Robust stability against plant model uncertainty

Robustness against plant uncertainties

So far, the AMB plant was assumed to be known accurately and uncertainty only affected the disturbance frequency. In practice, however, AMB systems are subject to various other uncertainties, for example due to inductance, bias currents, nominal position etc. It is usual practice to model these uncertainties as a lumped full block Δ and consider the control design problem for robustly stabilizing a set of plants given by $G := \{(I + \Delta_T W_T)G_0 : \|\Delta_T\|_\infty \leq 1\}$. A loop shaping design procedure to obtain robustly stabilizing controllers for a rotational speed variation of 0-10000 rpm, along with other model uncertainties can be found in [4]. Similar to what was done in [4, 5], all designs have been analyzed for robustness against uncertainties in the plant model. This is done according to the text-book procedure using a small gain argument on the transfer function seen by the uncertainty block, which is denoted by T. Figure 8 shows the resulting magnitude plots, using the weighting function W_T in [4]. The robustness margin decreases from a value of 0.57 to 0.20 from the nominal to robust designs. When considering uncertainty in the rotational speed (0-10000rpm) only, the gain of T is less than 1, confirming robust stability, as shown in Figure 9.

It is our perception that the robustness margins can be improved by incorporating plant model uncertainty into the design process. The robust synthesis procedure as used in this paper can be combined with standard μ-synthesis techniques leading to DK-like iterations.

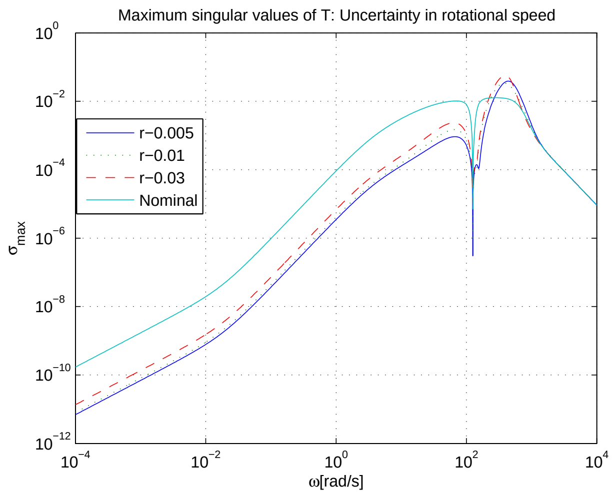

Figure 9: Robust stability against rotational speed uncertainty in plant

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References

[1] S. Boyd, El Ghaoui, Eric Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.

[2] S. G. Dietz, C. W. Scherer, and H. Köroğlu. Robust control against disturbance model uncertainty: a convex sollution. 46th IEEE Conference on Decision and Control, pages 842–847, New Orleans, USA., 2007.

[3] M. K. H. Fan, A. L. Tits, and J. C. Doyle. Robustness in the presence of mixed parameteric uncertainty and unmodeled dynamics. *IEEE Trans. Aut. Contr.*, 36(1):25–38, 1991.

[4] M. Fujita, K. Hatake, and F. Matsumura. Loop shaping based robust control of a magnetic bearing. *IEEE Contr. Syst. Mag.*, 13(4):57–65, August 1993.

[5] M. Fujita, K. Hatake, F. Matsumura, and K. Uchida. Experiments on the loop shaping based H_∞ control of magnetic bearing. Proc. American Control Conf., pages 8–12, San Francisco, CA., May 1993.

[6] R. Herzog, P. Bühler, and Conrad Gähler. Unbalance compensation using generalized notch filters in the multivariable feedback of magnetic bearings. *IEEE Trans. Control Systems Technology*, 4(5):580–586, September 1996.

[7] F. Matsumura, H. Kobayashi, and Y. Akiyama. Fundamental equation of horizontal shaft magnetic bearing and its control system design. *Trans. of IEE Japan*, 101(6):137–144, 1981.

[8] F. Matsumura and T. Yoshimoto. System modeling and control design of a horizontal-shaft magnetic-bearing system. *IEEE Trans. Magnetics*, (3):196–203, 1981.

[9] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. Aut. Contr.*, 42(6):819–830, 1997.

[10] A. Mohamed and I. Busch-Vishniac. Imbalance compensation and automation balancing in magnetic bearing systems using Q-Parameterization theory. *IEEE Trans. Control Systems Technology*, 4(5):484–493, September 1996.

[11] C. W. Scherer. Design of structured controllers with applications. Conference on Decision and Control, Sydney, 2000.

[12] C. W. Scherer. An efficient solution to multi-objective control problems with lmi objectives. *System and Control Letters*, 40(1):43–57, 2000.

[13] C. W. Scherer, P. Gahinet, and M. Chilali. Multiobjective output-feedback control via lmi optimization. *IEEE Trans. Aut. Contr.*, 42(7):896–911, 1997.

[14] C.W. Scherer. LMI relaxations in robust control. *European Journal of Control*, 12:3–29, 2006.

[15] J. D. Setiawan, R. Mukherjee, and E. H. Maslen. Synchronous sensor runout and unbalance compensation in active magnetic bearings using bias current excitation. *Journal of Dynamic Systems, Measurement and Control*, 124(14), March 2002.