Uncertainty Classification for Rotor-AMB Systems

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***Abstract*— In this paper, we adopt the gap metric tool to analyze uncertainty propagation in rotor-AMB systems. Of special interest is the effect of open loop plant uncertainties on the closed-loop stability. The results indicate that robustness property under general feedback varies with different sources of uncertainty, i.e., some uncertainties can be easily suppressed by any stabilizing controller, while other uncertainties can be amplified. As a result, uncertainty can be classified according to its effect on the closed-loop robustness. From design and analysis perspective, some uncertainties can be effectively approximated or even neglected while others must be accurately modeled, and addressed explicitly in the design and analysis.**

***Index Terms*— AMB, robustness, gap metric**

I. Introduction

For general rotating machinery applications such as turbomachinery, instability remains one of the most prevalent problems due to the presence of various uncertain destabilizing mechanisms in rotors and in the process dynamics [3], [2]. The deployment of magnetic bearings in these applications further exacerbates the problem. This is because magnetic bearings are open loop unstable, and additional uncertainties are introduced through the feedback loop: magnetic bearings, sensors, amplifiers and digital controllers. Indeed, it can be argued that stability robustness against uncertainty is the primary feedback design requirement for typical rotor-AMB systems.

Given the importance of the robustness, advanced robust optimal control synthesis techniques such as $\mathcal{H}_\infty$ and μ-synthesis methods are proposed to address the stability robustness in a systematic design procedure. To successfully apply these methods to the rotor-AMB systems, modeling, especially the uncertainty modeling, is of critical importance.

Unfortunately, physical uncertainties, by definition, defy exact mathematical characterization. Non-conservative estimation of uncertainty remains a challenging task in practice. There are several issues in the uncertainty specification: (1) not all uncertainties can be translated into the norm bounded uncertainty model without introducing extra conservativeness; (2) with the presence of many sources of uncertainty, the worst-case stability over the combination of uncertainties may not be realistic and can be overly conservative; (3) many uncertainties are real and have to be approximated by an enlarged complex counter part; (4), the phase information is lost when the uncertainties are characterized by a norm bound; and finally, (5) a large number of uncertainty specifications in robust control synthesis yields high order controllers which lack the transparency and are difficult to implement. To avoid these restrictions and reduce the conservatism, add hoc fixes to uncertainties are always employed. Yet, engineers need to understand the consequences of underestimating or overestimating uncertainty in the control design.

Compared to the rich research effort in robust control theory, few studies explore the dependence of the robust optimization to uncertainty specification. In this paper, we describe a systematic method developed to evaluate the uncertainty structure in rotor-AMB systems. Instead of quantifying each uncertainty for a specific rotor-AMB system, we explore various general uncertainty sources in AMB systems and their effects on optimization by using an analytical approach. The objective here is to obtain general answers to such questions as: (1) How does each uncertainty propagate through the feedback loop? (2) Which uncertainty is more difficult to tolerate by general feedback control? (3) Do we need to quantify all uncertainties or can we ignore certain perturbations? (4) What is the consequence if we underestimate an uncertainty bound?

II. Gap Metric and An SISO Rotor-AMB Model

To investigate uncertainty structure from feedback perspectives, we apply gap metric theory to an SISO rotor-AMB model. In this section, we present a brief overview of the gap metric, and describe the SISO rotor-AMB model used for robustness analysis.

A. Gap Metric

The gap metric and ν gap metric were originally introduced in [8] and [6], respectively. Both quantities are defined to characterize the ‘distance’ between the open loop dynamic systems (nominal and perturbed systems) by a scalar between 0 and 1. A small gap metric implies that the perturbed system is close to the nominal system from a feedback perspective even if the two open loop system dynamics are far apart. The ν gap metric is always less than or equal to the gap metric, i.e., the resulting analysis is tighter. Thus we adopt the ν gap metric as a tool for our robust analysis in this paper.

The ν gap metric is defined in [7] as

$$\delta_\nu(P_1,P_2) := \begin{cases} \|\tilde{G}_2 G_1\|_\infty, & \text{if } \det(G_2^* G_1)(j\omega)\neq 0 \\ \quad \forall\omega\in\mathbf{R} \text{ and } \mathrm{wno}(\det(G_2^*G_1))=0, & \\ 1, & \text{otherwise.} \end{cases} \tag{1}$$

where G_i and $\tilde{G}_i$, $i = 1, 2$, are the corresponding normalized right and left coprime factorization of the plant P_1 and P_2, $\text{wno}(g)$ denotes the winding number about the origin of $g(s)$ as s follows the standard Nyquist D-contour.

The ν gap metric is closely related to the robust stability of the closed loop system. To explore the relationship, we first introduce a so-called generalized stability margin:

$$b(P,C) = \begin{cases} \left\| \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \right\|_{\infty}^{-1}, \\ \qquad\qquad \text{if (P,C) is stable,} \\ 0, \text{ otherwise.} \end{cases} \tag{2}$$

where P and C denote the plant and the controller respectively. The generalized stability margin is defined in terms of the $\mathcal{H}_\infty$ norm of a 2×2 transfer function matrix.

With the notion of the generalized stability margin, the ν gap metric can be employed as a robustness measure of the open loop system against uncertainty from a closed-loop perspective. That is, if the nominal system has a general stability margin defined by b, then the stability margin of the perturbed system is related to uncertainty in terms of the gap metric by the triangular inequality

$$\begin{aligned} b(P_1, C_1) \geq \arcsin b(P_0, C_0) &- \arcsin \delta(P_0, P_1) \\ &- \arcsin \delta(C_0, C_1), \end{aligned} \tag{3}$$

as shown in [5]. With the identical controller, this result guarantees that the stability margin of the perturbed system would degrade by no more than $\delta(P_0, P_1)$. Therefore, if the plant gap metric is small, the perturbed system would also preserve certain stability margins comparable to the nominal system. Accordingly, system control design based on the nominal system with certain general stability margin is robust to uncertainties that generate small gap metrics.

The ν gap metric can be evaluated frequency wise as a distance between $P_1(j\omega)$ and $P_2(j\omega)$ in the form of

$$\delta_\nu(P_1, P_2) = \sup_{\omega} \frac{| P_2(j\omega) - P_1(j\omega) |}{\sqrt{1+ | P_1(j\omega) |^2)}\sqrt{1+ | P_2(j\omega) |^2}}. \tag{4}$$

Next, we will demonstrate how to use the ν gap metric for uncertainty analysis of rotor-AMB systems.

B. A SISO Rotor-AMB Model

Depending upon the applications of the model, rotor-AMB systems can be modeled at different levels of detail [4]. An SISO open loop transfer function is adopted in our evaluation of uncertainty,

$$\begin{aligned} P(s) \;\; = \;\; k_i \Bigg[& \underbrace{\frac{\varphi}{ms^2 - j\Omega g_0 s - k_x}} \\ & + \underbrace{\sum_{r=1}^{m} \frac{\eta_r}{ms^2 + (c_r - j\Omega g_r)s + k_r - \tau_r j}} \\ & + \underbrace{\sum_{k=1}^{n} \frac{\phi_k}{ms^2 + d_k(s) + k_k^2}} \Bigg] e^{-\theta s}, \end{aligned} \tag{5}$$

where the first part represents the rotor rigid body modes, the second and third parts are the flexible rotor and substructure model respectively. Parameters φ, ϕ, and η denote the corresponding mode shapes, m the modal mass, k_i the loop gain including the combination of sensor, amplifier and magnetic bearing gains, Ω spin speed, g_0 and g_r stand for the modal equivalent gyroscopic coupling, τ_r represents modal equivalent cross coupled stiffness or rotor viscous internal damping effect, i.e., $\tau = \Omega c_r$. d_k is the modal damping, and θ represents the time delay. Note that the AMB open loop stiffness k_x shows strong effect on the rigid body mode but very little effect on the flexible modes.

It is noted that the above SISO model characterizes certain key features of rotor-AMB systems. The coupling in two radial directions due to the cross coupled stiffness and gyroscopic effects is represented in the SISO transfer function by using complex notation. The other lateral cross talk between the two bearings can be decoupled by center of mass (C.G.) or similar coordinates if certain symmetric conditions are satisfied.

To simply the analysis and obtain analytical results, we adopt a single mode kernel of $P(s)$ as

$$P_m(s) = \frac{a + bj}{ms^2 + (c + gj)s + k + qj} e^{-\theta s}. \tag{6}$$

Coefficients a and b denote the real and imaginary part of the complex eigenvector while m, c, g, k and q represent the mass, damping, gyroscopic effect, stiffness as well as the cross coupled stiffness in modal form. We restrict the analysis to the case of $m > 0$, $c \geq 0$, $\theta > 0$. We note that the MDOF plant $P(s)$ can be viewed as a sum of a mode under investigation and a residue, i.e., $P(s) = P_m(s) + R(s)$. As we evaluate the gap metric frequency wise for each mode, the residual $R(s)$ is generally small. Hence, the gap metric based on a single mode second order system can be viewed as a reasonable approximation to the corresponding gap metric for MDOF systems.

III. Uncertainty Analysis

We evaluate the gap metric of the parametric uncertainty in each coefficient given in $P_m(s)$. Without loss of generality, we assume $m = 1$.

A. Gain and Mode Shape Uncertainty $(1+\varepsilon)a$ & $(1+\varepsilon)b$

We consider uncertainty in a as $a_r = (1+\varepsilon)a$, and denote the magnitude of the denominator of the transfer function (6) as ρ, i.e., $\rho = \sqrt{(-\omega^2 - g\omega + k)^2 + (c\omega + q)^2}$. Then the ν gap metric becomes

$$\delta_\nu = \sup_{\rho} \frac{\varepsilon a \rho}{\sqrt{(\rho^2 + a^2 + b^2)(\rho^2 + a^2(1+\varepsilon)^2 + b^2)}}.$$

The gap metric admits an analytical solution $\rho^4 = (a^2 + b^2)[a^2(1+\varepsilon)^2 + b^2]$, with

$$\delta_\nu = \frac{\varepsilon}{\sqrt{1 + \left(\frac{b}{a}\right)^2} + \sqrt{(1+\varepsilon)^2 + \left(\frac{b}{a}\right)^2}} \leq \frac{\varepsilon}{2+\varepsilon}. \tag{7}$$

The upper bound of $\varepsilon/(2+\varepsilon)$ is reached when b is zero.

Similarly, with perturbation $b_r = (1+\varepsilon)b$,

$$\delta_\nu = \frac{\varepsilon}{\sqrt{1+\left(\frac{a}{b}\right)^2} + \sqrt{(1+\varepsilon)^2+\left(\frac{a}{b}\right)^2}} \leq \frac{\varepsilon}{2+\varepsilon} \tag{8}$$

The upper bound of $\varepsilon/(2+\varepsilon)$ is reached when $a = 0$.

It is interesting that both parameter perturbations yield gap metrics bounded by the quantity $\varepsilon/(2+\varepsilon)$. This gap metric bound shows that parametric uncertainties in these coefficients are reduced to more than half of their open loop level when feedback is connected. For instance, if uncertainty ε is 50%, the maximum ν gap is less than 0.2. Recall that the triangular inequality (3) guarantees that the decrease in the perturbed system stability margin does not exceed the gap metric. Therefore, a closed-loop system designed based on nominal model with a stability margin larger than 0.2 remains stable even with a perturbation of 50%. Note that a reasonable general stability margin is between 1/5 to 1/3. This indicates that uncertainties in these coefficients are suppressed by the feedback.

The AMB current to the force relation is inherently nonlinear. The linearized current stiffness is an approximation. Error in k_i depends upon the bias level, the vibration amplitude and other factors such as the load and temperature. The gap metric result shows that the control design based on linearized AMB model still guarantees stability even with substantial amount of uncertainty under nonlinear and large orbit operation. The introduction of feedback essentially minimizes the AMB uncertainty. This benign property has been recognized by the successful application of linearized design in the field. As a result, the AMB uncertainties do not need to be addressed explicitly in the synthesis as long as certain general stability margin is specified for the feedback design.

Similarly, the amplifier and sensor may also exhibit certain variations. Dynamics of switching amplifiers can be highly nonlinear and frequency dependent. Sensor gains is closely related to the operating conditions such as the temperature, and they can also drift over time. However, the small gap metric to the gain variation indicates that a constant gain model to the amplifier and the sensor is usually adequate. The feedback can tolerate certain range of variations.

This also applies to uncertainty in mode shapes. From computational point of view, eigenvector is numerically less accurate than the eigenvalue computation. Experimentally, mode shape measurement will also be less accurate especially when complex mode shapes are involved. Fortunately, the gap metric result shows that mode shape uncertainties in both the rotor and the substructure are benign from feedback perspectives. Controller synthesis based on the nominal mode shape with certain stability margin would guarantee stability under the perturbed mode shape. Therefore, no uncertainty in the input matrix B and output matrices C need to be explicitly specified in the synthesis.

B. *Damping Uncertainty* $(1+\varepsilon)c$

Consider the parametric uncertainty $c_r = (1+\varepsilon)c$, we define ρ_1 and ρ_2 as $\rho_1 = \sqrt{(-\omega^2 - g\omega + k)^2 + (c\omega + q)^2}$ and $\rho_2 = \sqrt{(-\omega^2 - g\omega + k)^2 + [c(1+\varepsilon)\omega + q]^2}$. Then it is easy to verify that

$$\delta_\nu \leq \sup_\omega \frac{\varepsilon c\omega}{\rho_1 + \rho_2}.$$

The equality holds if $\rho_1\rho_2 = (a^2+b^2)^2$. We consider the two cases with different signs for the cross coupled stiffness q. First, if $q \geq 0$, then

$$\delta_\nu \leq \frac{\varepsilon}{\left|1+\frac{q}{c\omega}\right| + \left|(1+\varepsilon)+\frac{q}{c\omega}\right|} \leq \frac{\varepsilon}{2+\varepsilon}. \tag{9}$$

The upper bound of $\varepsilon/(2+\varepsilon)$ is reached when $q = 0$. Alternatively, if $q < 0$, it is easy to verify that when $c\omega \leq q \leq c(1+\varepsilon)\omega$, the ν gap metric is 1.0.

For substructure mode without external cross coupled stiffness, the gap metric to the viscous damping coefficient uncertainty is small. The result also applies to the hysteresis damping model. It is well known that damping is related to the energy dissipation. The energy dissipation mechanism is rather complex, and it can be highly nonlinear and frequency dependent [1]. Damping in practice is also difficult to measure and quantify. The assumed damping model, whether it is viscous or hysteresis, thus always involves some approximation. Fortunately, the gap metric indicates that the feedback stability is not sensitive to the damping model uncertainty. A controller design based on a simplified damping model such as an equivalent viscous model can be adequate.

This does not mean that the damping level is not important. On the contrary, the damping magnitude can be critical to the stabilization of flexible structural modes. As the structural damping level is rather small, the percentage deviation can be substantial. For instance, uncertainty in damping estimation can exceed 100% in some applications. With large magnitude deviation, the closed-loop stability can be compromised.

On the other hand, rotor internal damping is generally regarded as a destabilizing mechanism when rotor operates above the first bending frequency. Since the internal energy dissipation mechanism in the rotor structure is rather complex, often an equivalent viscous internal damping model is adopted. Parametric uncertainty associated with viscous damping coefficient accounts for the modeling errors. The viscous internal damping is modeled as a special case in the transfer function (6) by letting $q = -c\Omega$. With uncertainty in $c = (1+\varepsilon)c$, we define the amplitude of the denominator of the transfer function as ρ_1 and ρ_2, i.e., $\rho_1 = \sqrt{(-\omega^2 - g\omega + k)^2 + c^2(\omega-\Omega)^2}$ and $\rho_2 = \sqrt{(-\omega^2 - g\omega + k)^2 + c^2(1+\varepsilon)^2(\omega-\Omega)^2}$. Then the ν gap metric becomes

$$\delta_\nu \leq \sup_\omega \frac{\varepsilon c|\omega - \Omega|}{\rho_1 + \rho_2}. \tag{10}$$

The equality holds when $\rho_1\rho_2 = (a^2+b^2)^2$. Let $-\omega^2 - g\omega + k = 0$. We then have

$$\delta_\nu \leq \frac{\varepsilon}{2+\varepsilon}. \quad (11)$$

The gap metric upper bound of $\varepsilon/(2+\varepsilon)$ simply shows that the feedback stability is not particularly sensitive to uncertainty in rotor internal damping model. A feedback system with reasonable stability margin is robust to the certain amount of rotor viscous internal damping variation. This result certainly relieves the burden of the feedback designer. As it is well known, the internal damping mechanism is difficult to characterize.

In contrast, for rotor modes under the influence of cross coupled stiffness, uncertainty in damping can have great influence on stability. The cross coupled stiffness is a destabilizing mechanism. Both the cross coupled stiffness and the direct damping generate tangential forces with opposite directions. The whirling stability is determined by the balance of the two forces. Therefore, when the two forces are getting close, stability can be compromised. This condition is also revealed in the gap metric result. The gap is 1.0 when $c\omega \leq q \leq c(1+\varepsilon)\omega$.

C. *Gyroscopic Uncertainty* $(1+\varepsilon)g$

Consider uncertainty in g as $g_r = (1+\varepsilon)g$, we define ρ_1 and ρ_2 as, $\rho_1 = \sqrt{(-\omega^2 - g\omega + k)^2 + (c\omega+q)^2}$ and $\rho_2 = \sqrt{(-\omega^2 - g(1+\varepsilon)\omega + k)^2 + (c\omega+q)^2}$. Then we have

$$\delta_\nu \leq \sup_\omega \frac{\varepsilon|g|\omega}{\rho_1+\rho_2}.$$

The equality holds if $\rho_1\rho_2 = (a^2+b^2)^2$. When $c\omega+q=0$, the ν gap metric has maximum value of

$$\delta_{\nu max} = \frac{\varepsilon|g|\omega}{|-\omega^2 - g\omega + k| + |-\omega^2 - g(1+\varepsilon)\omega + k|}. \quad (12)$$

It is easy to verify that, when $(-\omega^2 - g\omega + k)[-\omega^2 - g(1+\varepsilon)\omega + k] \leq 0$, the ν gap could be as high as 1.0.

Gyroscopic effects are generally regarded as conservative and not destabilizing for rotors supported on mechanical bearings. However, AMB systems are not passive. Uncertainty in gyroscopic terms combined with AMB forces can compromise the closed-loop stability. This is mainly because that, with the variation of gyroscopic effects, rotor modes split into forward or backward modes with the modal frequency shifting. As a result, the phase characteristics at the shifted frequency can lead to instability.

D. *Stiffness Uncertainty* $(1+\varepsilon)k$

For uncertainty on $k_r = (1+\varepsilon)k$, we define ρ_1 and ρ_2 as $\rho_1 = \sqrt{(-\omega^2 - g\omega + k)^2 + (c\omega+q)^2}$ and $\rho_2 = \sqrt{(-\omega^2 - g\omega + k(1+\varepsilon))^2 + (c\omega+q)^2}$. It is easy to verify that

$$\delta_\nu \leq \sup_\omega \frac{\varepsilon|k|}{\rho_1+\rho_2}.$$

The equality holds if $\rho_1\rho_2 = (a^2+b^2)^2$. When $c\omega+q=0$, the ν gap metric reaches maximum value of

$$\delta_\nu \leq \frac{\varepsilon|g|\omega}{|-\omega^2 - g\omega + k| + |-\omega^2 - g\omega + k(1+\varepsilon)|}. \quad (13)$$

For the special case when $g=0$ and $k \leq 0$, $\delta_\nu \leq \varepsilon/(2+\varepsilon)$. It is easy to verify that when $(-\omega^2 - g\omega + k)[-\omega^2 - g\omega + k(1+\varepsilon)] \leq 0$, the ν gap is 1.0.

The result shows that the ν gap metric is small for the rotor rigid body mode with negative magnetic bearing open loop stiffness. This further illustrates that uncertainties in magnetic bearings can be handled well by stabilizing feedback. Therefore, a linearized bearing model is often adequate for the feedback design.

In contrast, the ν gap metric is large for uncertainty in the modal frequency for both rotor and substructure flexible modes with either viscous or hysteresis damping model. This indicates that with lightly damped substructure modal frequency uncertainty has to be explicitly addressed in the feedback design. The optimization using $\mathcal{H}_\infty$ with the objective of minimizing the general stability margin is not sufficient to guarantee the robustness. Intuitively, for lightly damped modes, a small perturbation in the modal frequency can generate large multiplicative uncertainty in the loop gain, and thus compromises the general stability margin. In addition, both the mode shape and damping affect the gap metric under the modal frequency uncertainty. Therefore, uncertainty of modal frequency should be handled individually in the synthesis for different modes.

E. *Cross Coupled Stiffness Uncertainty* $(1+\varepsilon)q$

For uncertainty on $q_r = (1+\varepsilon)q$, we define ρ_1 and ρ_2 as $\rho_1 = \sqrt{(-\omega^2 - g\omega + k)^2 + (c\omega+q)^2}$ and $\rho_2 = \sqrt{(-\omega^2 - g\omega + k)^2 + [c\omega + q(1+\varepsilon)]^2}$. It is easy to verify that

$$\delta_\nu \leq \sup_\omega \frac{\varepsilon|q|}{\rho_1+\rho_2}.$$

The equality holds if $\rho_1\rho_2 = (a^2+b^2)^2$. When $-\omega^2 - g\omega + k = 0$, the ν gap metric has maximum value of

$$\delta_\nu \leq \frac{\varepsilon|q|}{|c\omega+q| + |c\omega + q(1+\varepsilon)|}. \quad (14)$$

If $c \geq 0$ and $q \geq 0$, then

$$\delta_\nu \leq \frac{\varepsilon}{\frac{2c\omega}{q} + (2+\varepsilon)} \leq \frac{\varepsilon}{2+\varepsilon}. \quad (15)$$

Notice that if $q < 0$, it is easy to verify that the ν gap metric can be as high as 1.0 when the condition $(c\omega+q)[c\omega+q(1+\varepsilon)] \leq 0$ holds.

The presence of cross coupled stiffness in seals, hydrodynamic bearings, turbine and pump impellers is widely recognized as a major destabilizing mechanism in turbomachinery applications. Depending upon the sign of q, the cross coupled stiffness stabilizes either the backward or forward mode while destabilizing the other one. It is expected that the gap metric to uncertainty in q is large. The result simply confirms that a feedback system with certain general stability margin could not guarantee to be stable in the presence of cross coupled stiffness uncertainty. This uncertainty must be addressed explicitly in the control design.

F. Time Delay Uncertainty

Uncertainty in time delay comes from the A/D, D/A converters and the computational delay. It is assumed that the sampling rate is high so that the delay term is relatively small for AMB systems. With perturbation on $\theta = (1+\varepsilon)$, and a open loop plant $P = P_0 e^{-\theta s}$, the ν gap metric can be expressed as

$$\delta_\nu = \sup_\omega \frac{|1 - e^{-j\omega\varepsilon\theta}|}{|P_0(j\omega)| + \frac{1}{|P_0(j\omega)|}}. \tag{16}$$

Let ω_c be the cross over frequency, i.e., $|P_0(\omega_c)| = 1$, and further assume that the delay term is small compared to the cross over frequency, i.e., $\omega_c \leq 1/\theta$, then the gap metric can be approximated by

$$\delta_\nu \leq \frac{\varepsilon\theta\omega_c}{2} \leq \frac{\varepsilon}{2}. \tag{17}$$

This gap metric upper bound reveals that for a small time delay incurred in the digital implementation, uncertainty in the time delay can be suppressed by the feedback. No additional time delay uncertainty is required if certain generalized stability margin is maintained.

G. Uncertainty Classification

From modeling and identification perspectives, uncertainties in rotor-AMB systems can be better quantified in either physical or modal space for each component. The effect of the resulting uncertainty to the closed-loop stability under any stabilizing controller are revealed by the corresponding gap metric numbers. While some of the uncertainties with large gap metrics are critical and must be retained in the synthesis to avoid the fragile behavior, other uncertainties are easily tolerated by any stabilizing feedback controller, and therefore can be ignored in the control synthesis. According to the gap metric, typical rotor-AMB uncertainties are summarized in Table I.

Components	Uncertainty	ν Gap Metric
Rotor	Natural Frequency	Can be high
	Internal Damping	$\frac{\varepsilon}{2+\varepsilon}$
	Mode Shape	$\frac{\varepsilon}{2+\varepsilon}$
	Gyroscopics	Can be high
AMB	Open loop k_x	$\frac{\varepsilon}{2+\varepsilon}$
	Current k_i	$\frac{\varepsilon}{2+\varepsilon}$
Substructure	Natural Frequency	Can be high
	Damping	$\frac{\varepsilon}{2+\varepsilon}$
	Mode Shape	$\frac{\varepsilon}{2+\varepsilon}$
Sen/AMP/Filter	Gain Error	$\frac{\varepsilon}{2+\varepsilon}$
Digital Controller	Discretization Error	$\frac{\varepsilon}{2+\varepsilon}$
	Time Delay	$\frac{\varepsilon}{2}$
Process	Cross Coupled Stiffness	Can be high

TABLE I

ν GAP METRIC CLASSIFICATION OF ROTOR-AMB SYSTEM UNCERTAINTY.

IV. APPLICATIONS

We apply the gap metric results to an AMB test rig. Our rotor-AMB control test rig was built as a platform for investigating different control schemes. The test rig consists of a rotor bearing assembly, a compliant foundation and electronic control systems as reported in [4].

We compute the ν gap metric for a 5% perturbation of the three rotor bending modes. The ν gap metric numbers of the first and second rotor modes are 0.979 and 0.907 respectively. This indicates that the corresponding natural frequency uncertainties are difficult to handle by a typical loopshaping $\mathcal{H}_\infty$ controller which aims at optimizing the general stability margin. To achieve the robustness to the natural frequency perturbation, uncertainty has to be taken into account explicitly in the synthesis. One approach is to adopt μ-synthesis. Compared to the first two modes, the ν gap metric of the third mode is only 0.046. This guarantees that the corresponding uncertainty can be easily handled by any stabilizing controller with certain general stability margin. In fact, controller design based on only the first two rotor modes were performed and tested successfully. The result shows that the third mode has little effect in stability. The gap metric analysis can always be used as a good screening tool.

A. Substructure

The gap metric analysis result is also verified by the stabilization of substructure modes. The structured model was identified by using the standard prediction error/maximum likelihood method [4]. The resulting MIMO substructure model contains ten modes. We then compute the ν gap metric of each substructure mode with 5% perturbation. The results are listed in Table II. The gap metric values for the mode No.6-No.9 are quite small compared to other modes. This suggests that we may be able to ignore the uncertainties in these four modes. Indeed, the μ-synthesis without modal frequency uncertainties in these four modes stabilizes the rotor. This greatly simplifies the synthesis and the resulting controller order is also reduced.

Substructure	Frequency (Hz)	ν Gap
mode 1	116.11	0.4925
mode 2	134.76	0.6650
mode 3	138.34	0.8965
mode 4	158.27	0.2871
mode 5	183.60	0.1035
mode 6	225.70	0.0347
mode 7	275.39	0.0422
mode 8	290.46	0.0476
mode 9	299.02	0.0480
mode 10	356.13	0.1833

TABLE II

GAP METRIC OF ROTOR MODAL FREQUENCY UNCERTAINTY.

In contrast, the result indicates that modes No. 1-No. 5 are critical. The closed-loop can be sensitive to these mode uncertainties if they are not specified properly in the control design. This is illustrated by a test rig instability incidence. After a few years of operation, a μ-controller with a reasonable stability margin (sensitivity peak less than 3.0) became unstable during testing. This controller

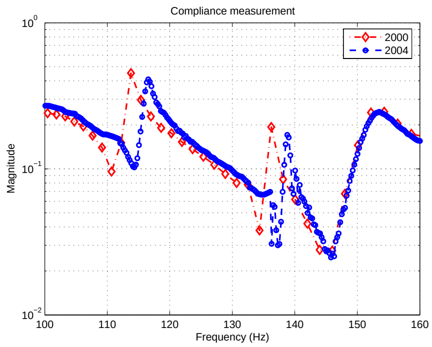

Fig. 1. Compliance measurement comparison.

was designed based on 2% modal frequency uncertainty for the substructure mode. A careful examination of substructure compliance measurements taken in 2000 and again in 2004 revealed a small shifting of the substructure modes. Fig. 1 shows that modes at 113 Hz and 134 Hz shift about 2%. This mode variation was in part due to reassembly of the rig. Although the shift in modal frequency is small, the change in the closed-loop stability is dramatic given that the controller has reasonable stability margin.

B. Magnetic Bearings

Finally, we exam the system robustness by testing the sensitivity function under different AMB bias levels. The gap metric results show that a feedback design with a reasonable stability margin can tolerate certain level of AMB uncertainty. The current and open loop stiffness k_i and k_x are linearized based on specific bias level, and the resulting k_i is linearly related to the bias while k_x is a quadratic function of the bias. To evaluate the robustness for a particular controller designed at 2.75 ampere, we increased the AMB bias level from 2.75 amperes to 3.67 amperes, and measured the sensitivity function at both two bias levels. With the increase of bias the actuator gain k_i increased 33% and open loop stiffness increased more than 78%. At bias level of 2.75 amperes, the maximum diagonal sensitivity function shown in Fig. 2 reveals a peak of 2.2.

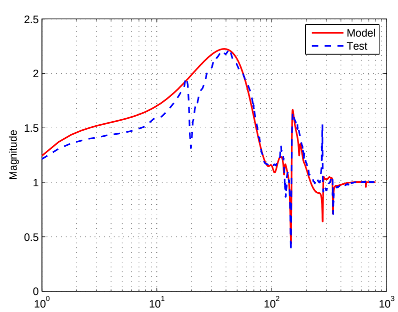

Fig. 2. Sensitivity function with bias of 2.75 A.

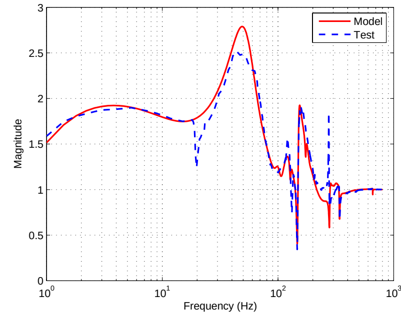

Fig. 3. Sensitivity function with bias of 3.67 A.

This indicates a decent stability margin. When the current was increased to 3.67 amperes, the system remained stable and the measured sensitivity function is shown in Fig. 3. The measured maximum diagonal sensitivity peak is 2.52. The increase of bias does degrades the stability margin slightly. However, the result demonstrates that the AMB uncertainty can be effectively suppressed by feedback. Note that the tested controller was designed based on a mixed sensitivity $\mathcal{H}_\infty$ performance. No AMB uncertainties were incorporated into the design.

V. Conclusions

In this paper, we applied the ν gap metric to analyze uncertainty propagation for rotor-AMB systems. Analytical results are obtained for a SISO single mode model. The gap metric result indicates that some uncertainties are suppressed while others are amplified through feedback. Consequently, uncertainties can be classified according to their effects on the closed loop system stability. While some uncertainties must be explicitly addressed and precisely quantified in the control synthesis, other uncertainties can be simplified or even neglected in the synthesis. Finally, we demonstrated the application of the gap metric on an AMB test rig. Both the analysis and experimental results show that gap metric is an important tool in the design and analysis of rotor-AMB systems.

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